Needham, Chapter 2, Exercise 11, p. 113.
G. Palmer, July 19, 2020, 11:10 am PST

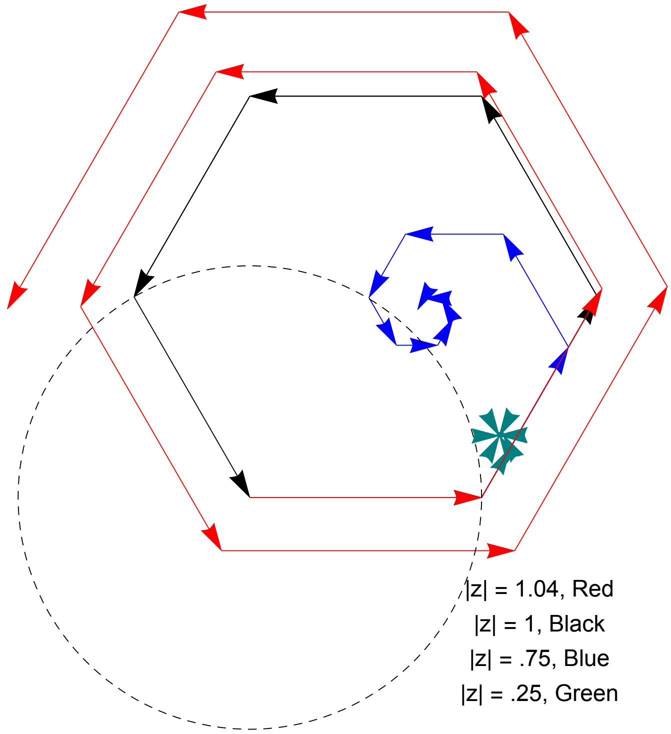


Figure 1. $\sum_{n=0}^{10} z^n$

11. Show that each of the following series has the unit circle as its circle of convergence, then investigate the convergence on the unit circle. You can guess the correct answers by "drawing the series" in the manner of [18] on p. 80.

The radius of convergence R can be found using the equations on pp. 74-75.

(i) $\sum_{n=0}^{\infty} z^n$

We could write $c_n = c_{n+1} = 1$. Then

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{1}{1} \right| = 1$$

which shows that the circle of convergence is the unit circle. We can also see that the series does not converge for z on the unit circle.

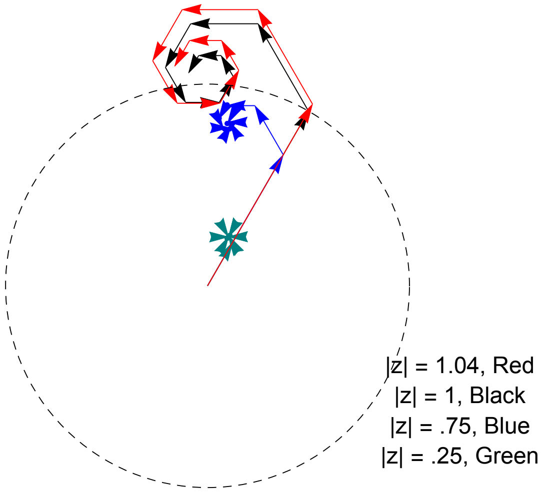
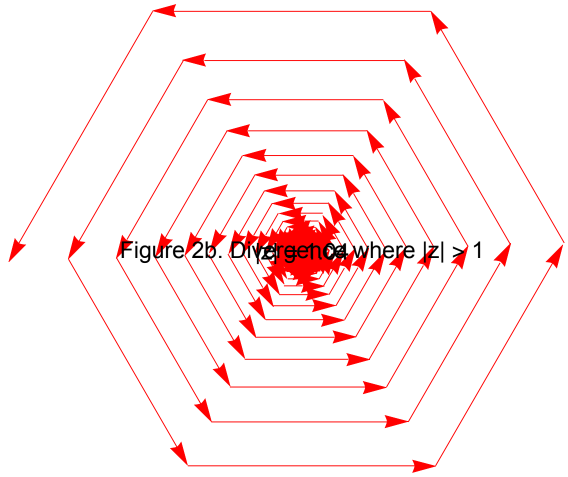


Figure 2a. $\sum_{n=1}^{10} z^n/n$



(ii) $\sum_{n=1}^{\infty} \frac{z^n}{n}$ [This series is –Log[1–z].]

Ratio test:

$$\lim_{n\to\infty} \left| \frac{1/n}{1/(n+1)} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \right| = \lim_{n\to\infty} \left| 1 + \frac{1}{n} \right| = 1$$

So the radius of convergence is 1 and the sum converges everywhere on the disc (Figure 2a, non-red arrows), but the sum diverges outside the disc (Figure 2b). The arrow here in each series is the length of |z|. because the series begins at n = 1. The red arrows indicate |z| = 1.04, which is outside the disc. In 2a, it is hard to tell whether the sum is converging or diverging. By plotting 100 iterations rather than 10, we can see the divergence in 2b. For z on the boundary, the series converges.

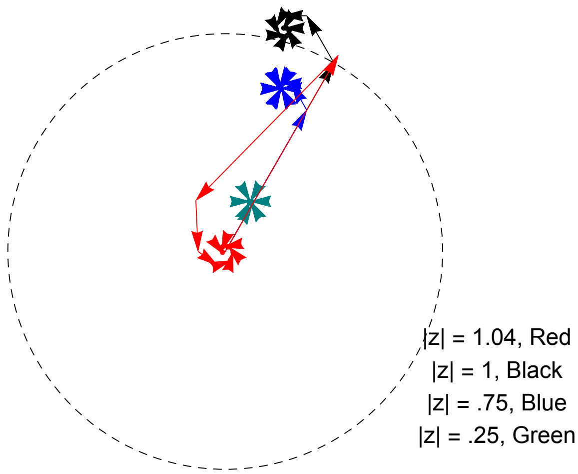


Figure 3a. $\sum_{n=1}^{10} z^n/n^2$

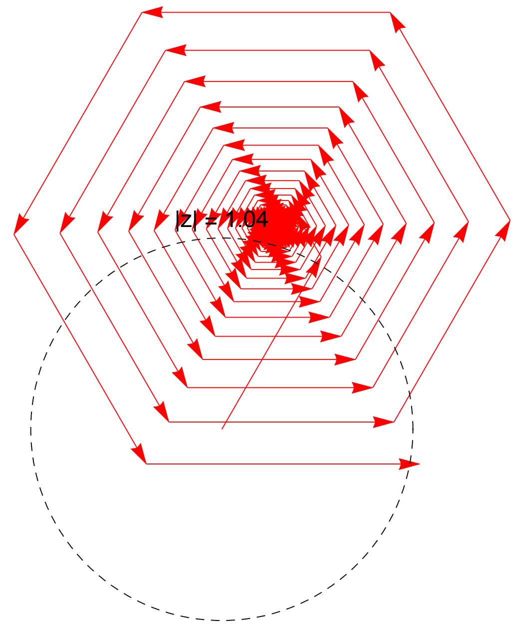


Figure 3b. Divergence where |z| > 1

(iii) $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$

$$R = \underset{n\to\infty}{\text{Lim}} \left| \frac{n^2}{(n+1)^2} \right| = \underset{n\to\infty}{\text{Lim}} \left| -\frac{2}{n+1} + \frac{1}{(n+1)^2} + 1 \right| = 1$$

The sum converges everywhere on the unit disc. It appears to also converge outside the unit disc as indicated by the red arrows in Figure 3a, but taking the interations out to 200 reveals the failure to converge in 3b. For z on the boundary, the series converges.

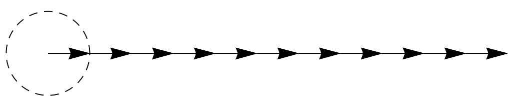
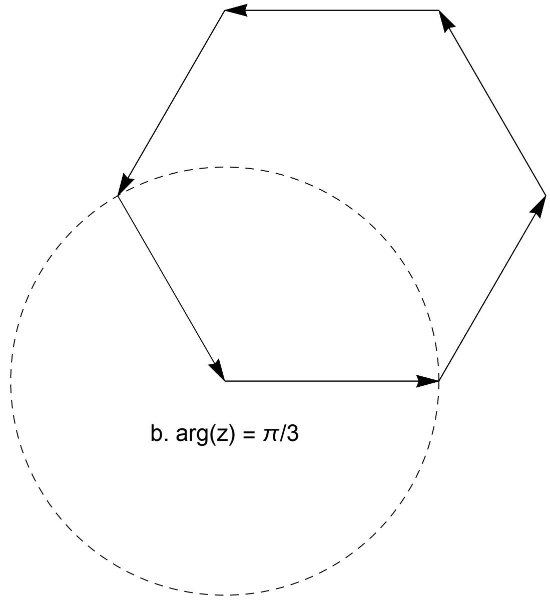
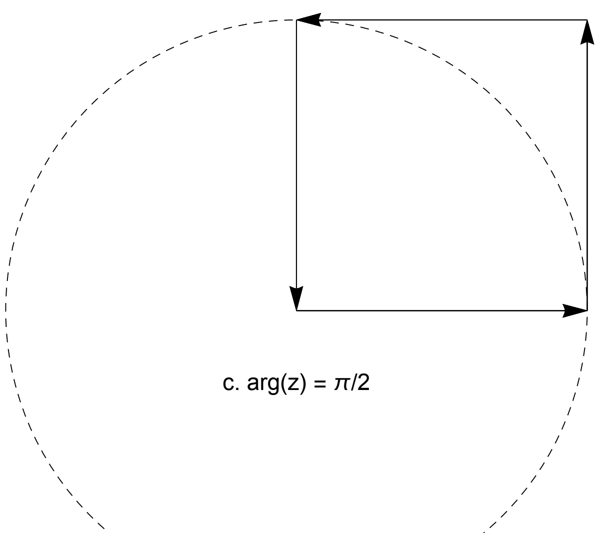


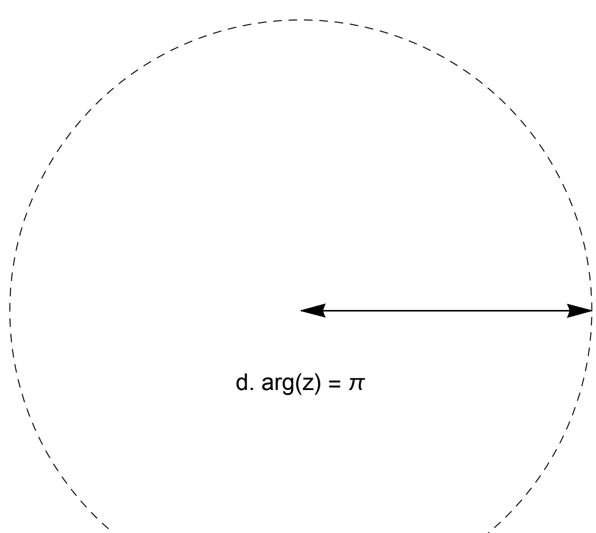
Figure 4a. $\sum_{n=0}^{10} z^n$, arg(z) = 0



b. arg(z) = $\pi/3$



c. arg(z) = $\pi/2$



d. arg(z) = π

On the boundary series (i) fails to converge for any of the chosen initial values of z. We suspect it does not converge for initial z anywhere on the boundary.

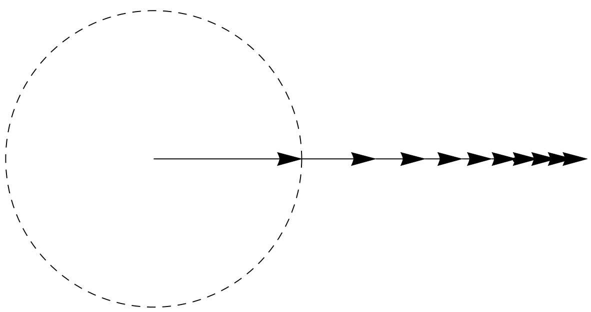
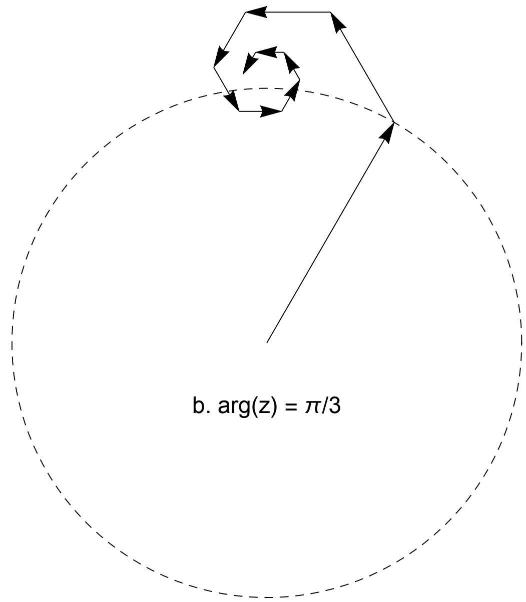
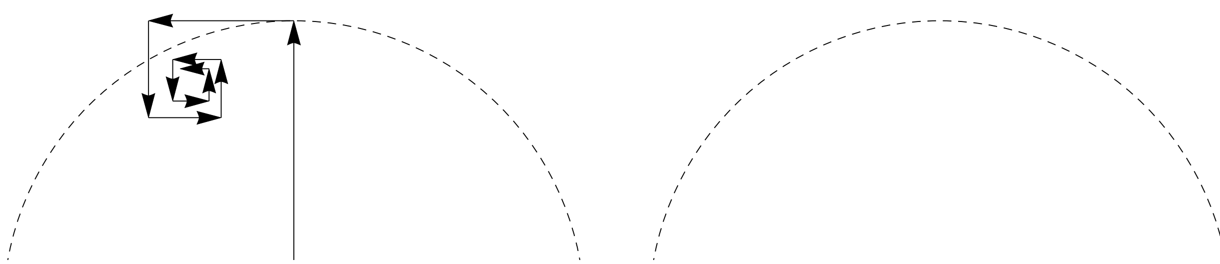
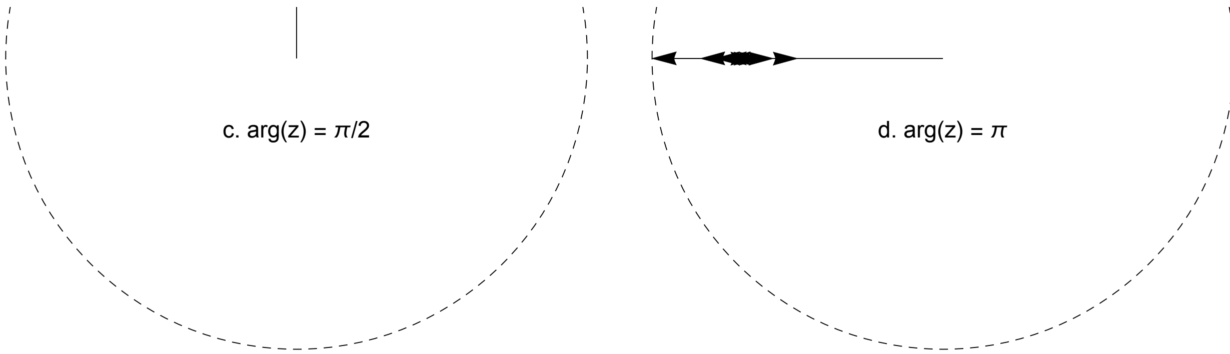


Figure 5a. $\sum_{n=0}^{10} z^n/n$, arg(z) = 0



b. arg(z) = $\pi/3$





The series (ii) clearly converges where arg(z) = $\pi/3$ and $\pi/2$, but one can not tell from the plots for the other initial positions of z on the boundary whether they converge.

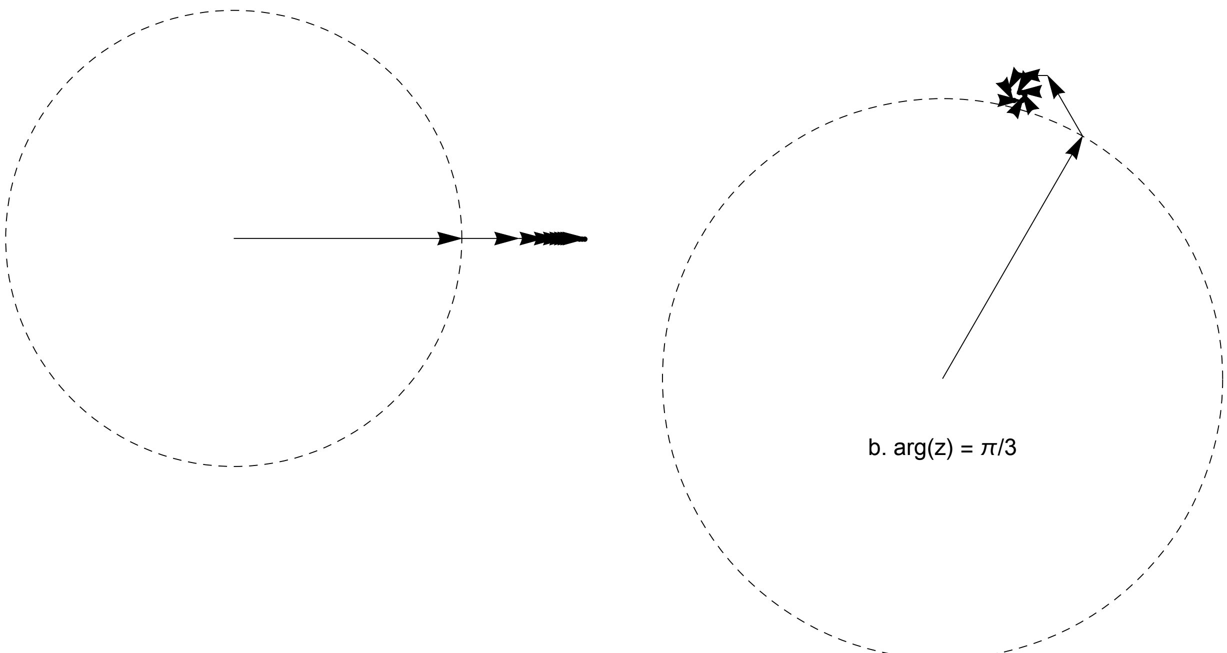
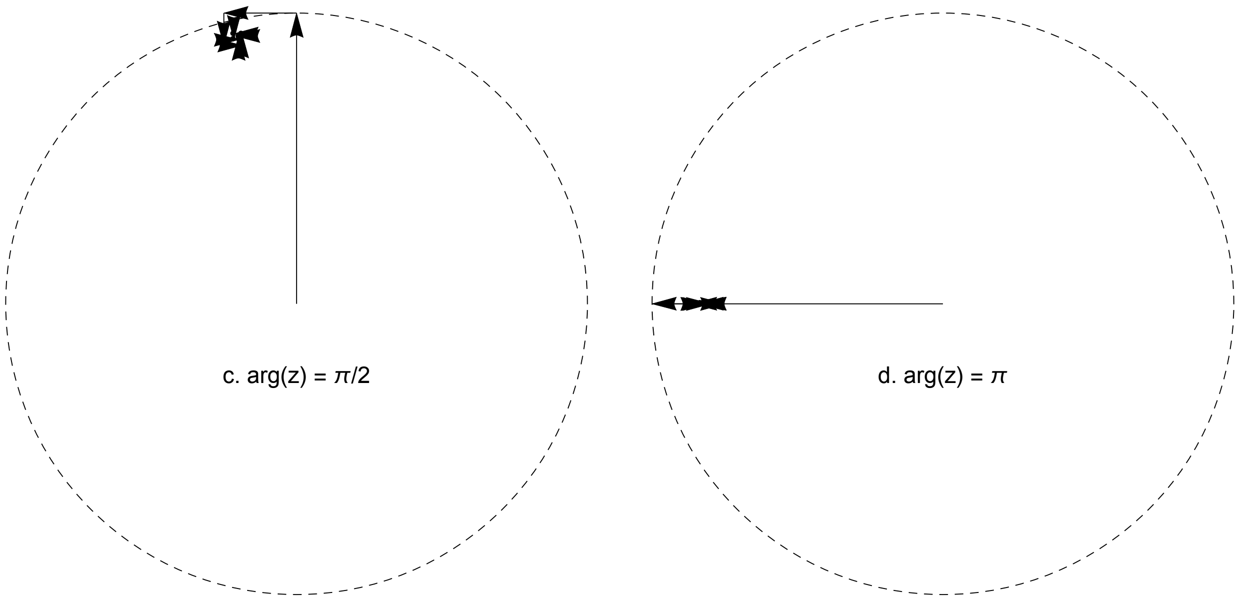


Figure 6a. $\sum_{n=0}^{10} z^n/n^2$, arg(z) = 0



Series (iii) $\sum_{n=0}^{10} z^n \big/ n^2$ clearly converges where arg(z) = $\pi/3$ and $\pi/2$. It may also converge at the other points, but one can not tell from the plots.