Needham, Chapter 2, Exercise 10, p. 113.
G. Palmer, July 9, 2020, 8 am PST

(10) Reconsider the series (18) on p. 77.

$$\frac{1}{1+x^2} = \sum_{j=0}^{\infty}\left[\frac{\sin(j+1)\,\phi}{\left(\sqrt{1+k^2}\right)^{j+1}}\right]X^j, \text{ where } X = (x-k)$$

(i) Show that we recover the correct series (missing the odd powers of x) when the centre of k of the series is at the origin.

When the centre is at the origin:

$\phi = \text{Arg}[i-k] = \text{Arg}[i] = \pi/2$ (p. 77)
$k = 0$

$$= \sum_{j=0}^{\infty}[\sin[(j+1)\,\pi/2]\,(x-k)^j$$

$$= \sum_{j=0}^{\infty}[\sin[(j+1)\,\pi/2]\,x^j$$

$$= \sum_{j=0}^{\infty}[\cos[j\pi/2]\,x^j$$

$$= \sum_{n=0}^{\infty}[\cos[(2\,n+1)\,\pi/2]\,x^{2\,n+1} + \cos[2\,n\,\pi/2]\,x^{2\,n}]$$ (by separating the even and odd powers of x)

$$= \sum_{n=0}^{\infty}[\cos[n\pi + \pi/2]\,x^{2\,n+1} + \cos[n\,\pi]\,x^{2\,n}]$$

$$= \sum_{n=0}^{\infty}[-\sin[n\pi]\,x^{2\,n+1} + (-1)^n\,x^{2\,n}]$$

$$= \sum_{n=0}^{\infty}[0 + (-1)^n\,x^{2\,n}]$$

$$\frac{1}{1+x^2} = \sum_{j=0}^{\infty}[0 + (-1)^j\,x^{2\,j}] = \sum_{j=0}^{\infty}(-1)^j\,x^{2\,j}$$ substitute j for n

Which is the series presented on p. 64.

(ii) Find a value of k such that the series is missing all the powers X^n, where n = 2, 5, 8, 11, 14, ... Check your answer using a computer.

$$\frac{1}{1+x^2} = \sum_{j=0}^{\infty}\left[\frac{sin[(j+1)\,\phi]}{\left(\sqrt{1+k^2}\right)^{j+1}}\right] X^j, \text{ where X = (x-k)}$$

j = 0 $\quad \frac{sin[(j+1)\,\phi]}{\left(\sqrt{1+k^2}\right)^{j+1}}\,(x-k)^0 = \frac{sin\,\phi}{\left(\sqrt{1+k^2}\right)^{j+1}}$

j = 1 $\quad \frac{sin[(1+1)\,\phi]}{\left(\sqrt{1+k^2}\right)^{j+1}}\,(x-k) = \frac{sin\,(2\,\phi)}{\left(\sqrt{1+k^2}\right)^{j+1}}$(x-k)

j = 2 $\quad \frac{sin[(1+1)\,\phi]}{\left(\sqrt{1+k^2}\right)^{j+1}}\,(x-k)^2 = \frac{sin\,(3\,\phi)}{\left(\sqrt{1+k^2}\right)^{j+1}}\,(x-k)^2 \;= 0$

ϕ is arg(i-k) = $\tan^{-1}(-1/k)$. Then

$$\frac{sin\left(3\,tan^{-1}(-1/k)\right)}{\left(\sqrt{1+k^2}\right)^{j+1}}\,(x-k)^2 \;= 0$$

The equation is true whenever $sin\left(3\,n\,tan^{-1}(-1/k)\right) = 0$, that is, whenever 3 n ϕ = n π, so $\phi = \pi/3$. Then

$\pi/3 = \tan^{-1}(-1/k)$

$-1/k = \tan(\pi/3)$

$k = -1/\tan(\pi/3) = -1/\sqrt{3}$

Check your answer using a computer.

The check involves calculating the first 15 terms of the series. Every third term is 0, showing that the series is missing all the powers X^n, where n = 2, 5, 8, 11, 14. The powers of (x-k) are shown in the results for the non-zero terms.

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In[11]:= Clear["Global`*"];

series[k_, ϕ_] :=
  Table[(Sin[(j + 1) ϕ] / Sqrt[(1 + k^2)]^(j + 1)) (x - k)^j, {j, 0, 14}];

series[-1 / Sqrt[3], Pi / 3]
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$$\left\{\frac{3}{4}, \frac{3}{8}\sqrt{3}\left(x+\frac{1}{\sqrt{3}}\right), 0, -\frac{9}{32}\sqrt{3}\left(x+\frac{1}{\sqrt{3}}\right)^3, -\frac{27}{64}\left(x+\frac{1}{\sqrt{3}}\right)^4, 0, \frac{81}{256}\left(x+\frac{1}{\sqrt{3}}\right)^6, \frac{81}{512}\sqrt{3}\left(x+\frac{1}{\sqrt{3}}\right)^7,\right.$$

$$\left. 0, -\frac{243\sqrt{3}\left(x+\frac{1}{\sqrt{3}}\right)^9}{2048}, -\frac{729\left(x+\frac{1}{\sqrt{3}}\right)^{10}}{4096}, 0, \frac{2187\left(x+\frac{1}{\sqrt{3}}\right)^{12}}{16\,384}, \frac{2187\sqrt{3}\left(x+\frac{1}{\sqrt{3}}\right)^{13}}{32\,768}, 0\right\}$$