Needham, Chapter 2, Exercise 1
G. Palmer, May 19, 2020, 11:30 am

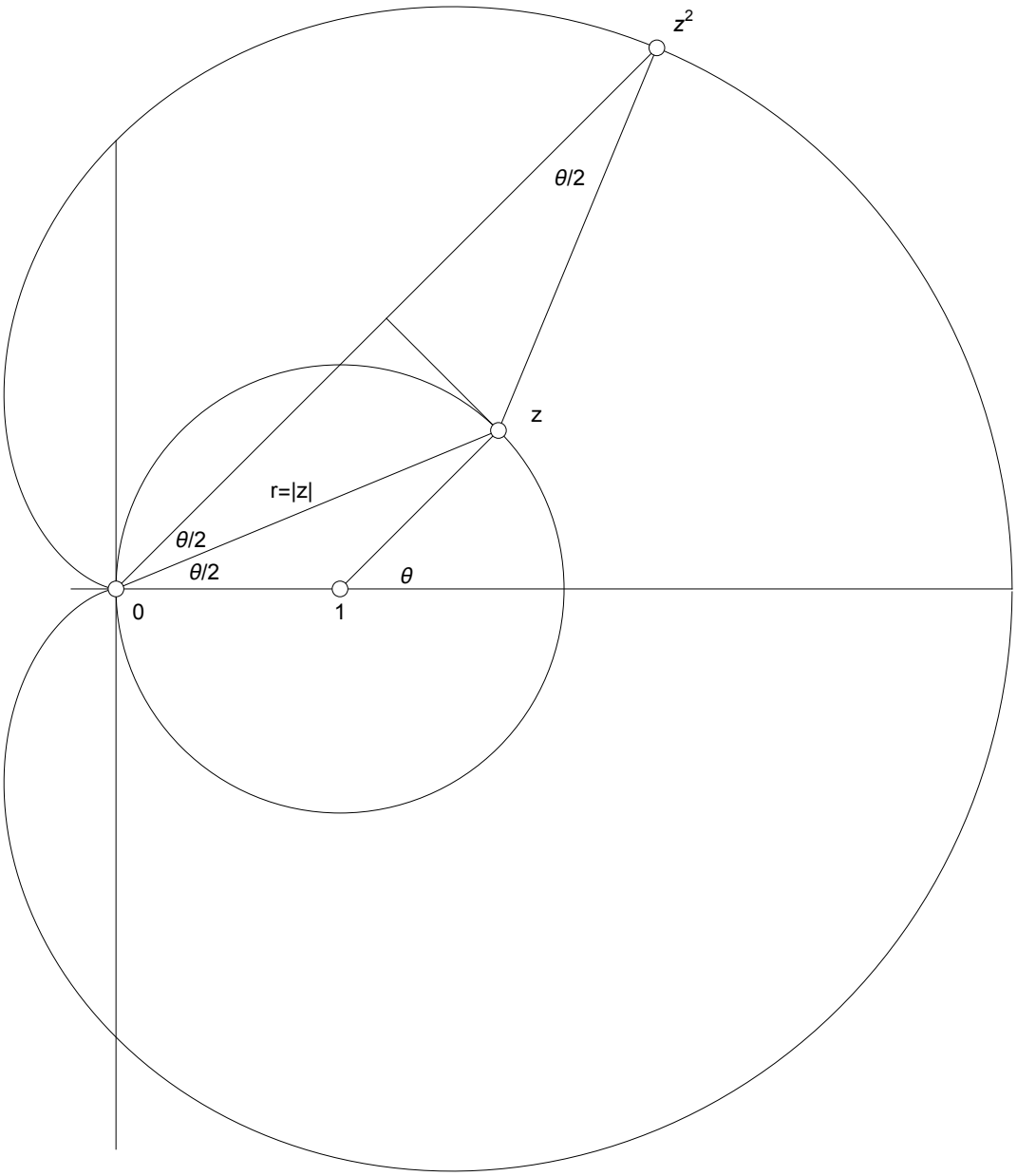

Figure 1. Cardioid

1 Sketch the circle |z-1| = 1. Find (geometrically) the polar equation of the image of this circle under the mapping $z \longrightarrow z^2$. Sketch this image curve, which is called a cardioid.

The circle of radius 1 with center at 1 is $z = (1 + e^{i\theta})$ or $z = (1 + \cos\theta + i \sin\theta)$. This circle is centred at 1. Figure 1 shows the angle of z at 1 is θ and the angle of z at 0 is $\theta/2$. Then z^2 must have the

angle θ at 0, which makes z^2 parallel to (z-1). Then, we want to know if $|z^2|/2$ equals $|z| \cos \theta/2$, that is, if $r^2/2 = r \cos \theta/2$, where r = 2 cos θ/2.

$$(1/2)\ 4 \cos^2(\theta / 2) = 2 \cos \theta/2 \cos \theta/2$$

$$2 \cos^2 (\theta/2) = 2 \cos^2 (\theta/2)$$

Therefore, the perpendicular from z to the line from 0 to z^2 is the bisector of z^2, so the triangle 0-z-z^2 is isosceles with angles as shown.

$$r^2 = 4 \cos^2(\theta / 2)$$

Then the trig formula for the cardioid must be

$$w(\theta) = 4 \cos^2(\theta / 2)\ (\cos \theta + i \sin \theta)$$

which can be written in polar coordinates as

$$w(\theta) = (4 \cos^2(\theta/2)\ , \theta) \qquad (1)$$

$$= (2\ (1 + \cos \theta), \theta) \quad \text{by standard trig identity}$$

The formula in (1) was used to plot the cardioid shown in Figure 1.

Illustration of a one-to-one mapping of the circle to the cardioid done as a recreation.

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