**Plotting streamlines, pp. 527-536.**
**G. Palmer, April 28, 2019, 3 pm PST**

This is about what I perceive to be a difficulty in understanding and plotting the complex potential or its streamlines and potentials. A look at the definitions shows that the complex potential $\Omega(z) = \Phi + i\Psi$ maps streamlines to horizontal lines. This discussion will focus on the streamlines or $\Psi$-lines.

Ch 11, III, 5 The Complex Potential, p. 501

"The complex potential maps streamlines to horizontal lines and equipotentials to vertical lines." This is illustrated in [21].

Ch 12, V, 1, p. 527

"complex potential $\Omega$ of this disturbed flow as a mapping … maps B [a boundary] to a horizontal line.

"The problem of finding flows around a given B therefore amounts to finding conformal mappings $\Omega$ with this property."

It is clear that $\Omega$ (z) maps streamlines to horizontal lines, but when we first plot the streamlines in the z-plane, we will most likely plot them as a contour plot using $\Psi$, which is passed to the plotting function as a series of constants (ContourPlot[$\Psi$(x,y) = {$c_1$, $c_2$, ...}]). This gives the impression, to me at least, of contradicting the definitions, but it does provide a practical method of producing the plots we see in Needham. We try to improve our understanding of why there is actually no contradiction.

Ch 12, V, 2, p. 527 bottom

Example: $\Omega$ (z) = z + 1/z

This is illustrated by [12], p. 538. Figure 1 as a facsimile of [12]. Figure 1 uses $\Psi(x, y) = \mathrm{Im}[\Omega(z)]$ in the contour plot function.

Ch 12, V, 2, p. 530

The same $\Omega$ is illustrated in more detail in [15], accompanied by 4th bullet above "so we have a one-to-one mapping (with inverse $\Omega^{-1}$)…"

Out[731]=

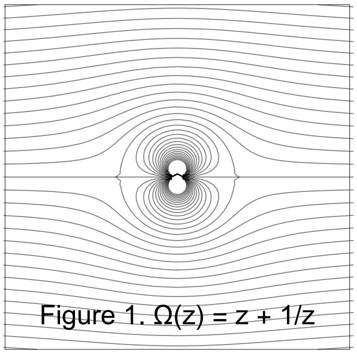

Figure 1. Ω(z) = z + 1/z

We can continue in the same vein.

Ch 12, V, 3, p. 532

A unit disc is inserted into the flow of a dipole, which has (undisturbed) complex potential

$$\Omega_u(\mathrm{z}) = \frac{(1+i)}{(z-2)}$$

It is illustrated in [17].

A source has an undisturbed complex potential

$$\Omega_u(\mathrm{z}) = \log(\mathrm{z}\text{-}2\text{-}1)$$

It is illustrated in [18]. The disturbed flow (combined flow of conjugate sources) is given as

$$\Omega_d(\mathrm{z}) = \log(\mathrm{z}\text{-}2\text{-i}) + \log(\mathrm{z}\text{-}2\text{+i})$$

with the instruction “Use a computer to verify that this formula produces [19].” We are able to verify it with the contour plot function by plotting $\Psi(\mathrm{z}) = \mathrm{Im}(\Omega_d(\mathrm{z}))$ for several values of $\Psi$ (Figure 2).

Out[751]=

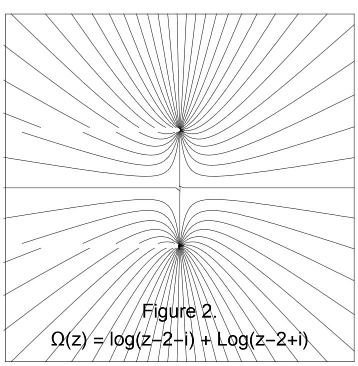

Figure 2.
Ω(z) = log(z–2–i) + Log(z–2+i)

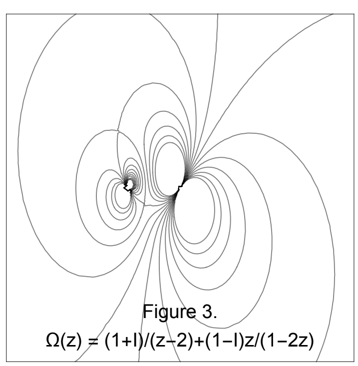

Figure 3.
Ω(z) = (1+I)/(z–2)+(1–I)z/(1–2z)

On p. 536, we return to the dipole inversions with

$$\Omega_d(z) = \Omega_u(z) + \Omega_u^{\dagger}(z) = \frac{(1+i)}{z-2} + \frac{(1-i)\,z}{1-2\,z}$$

and the suggestion to “use a computer to verify that the formula does yield the flow in [20].” We are able to verify it in figure 3 using the same method used to plot figure 2.

In summary. In all cases we use the contour plot function using Ψ(z) = Im(Ω(z)) to produce the streamlines in the z-plane as shown in Needham’s plots. One can then apply w = Ω(z) to all the points of any streamline to produce a horizontal line of the grid. The difference k between the Ψ values of any two lines of the grid equals the difference k between the Ψ values of any two streamlines in the z-plane.

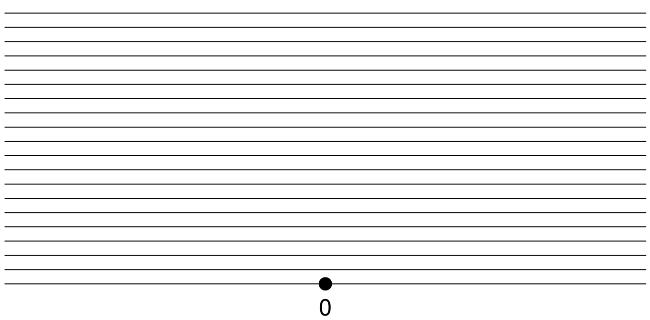


Figure 4. Horizontal streamlines k = .15

Out[772]=

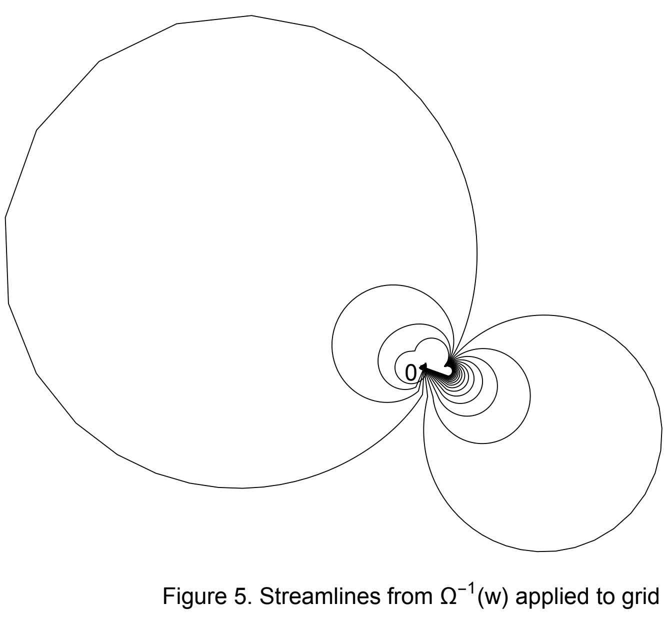


Figure 5. Streamlines from $\Omega^{-1}$(w) applied to grid

There is a second way to produce the streamlines in the z-plane. They can be produced from the grid in the w-plane by plotting $z = \Omega^{-1}(w)$. Figure 5 is produced this way from the horizontal lines in Figure 4 using the inverse of

$$\Omega_d(z) \;=\; \Omega_u(z) + \Omega_u^{\dagger}(z) = \frac{(1+i)}{z-2} + \frac{(1-i)\,z}{1-2\,z}$$

The inverse is

$$\Omega^{-1}(w) = (1/2)\ (5w + 4 + \sqrt{9\,w^2 + 24\,w + 8}\,)/(2w+(1-i))$$

Figure 5 bears some similarity to figure 3. With more skill in plotting continuations of complex inverse functions, one might be able to approximate figure 3 more closely, but I think it demonstrates the principle.

Out[780]=

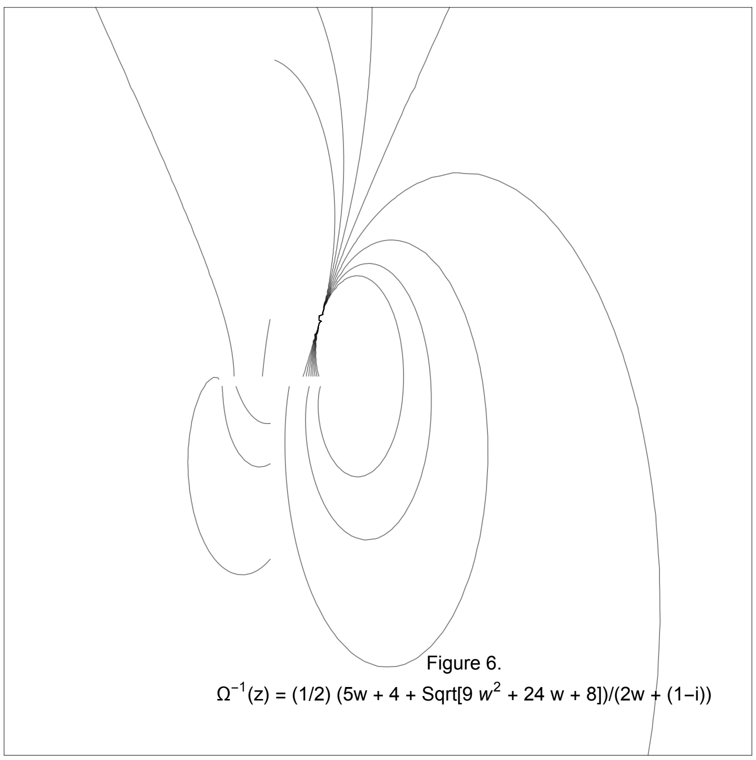

Figure 6.
$\Omega^{-1}(z) = (1/2)\ (5w + 4 + \mathrm{Sqrt}[9\,w^2 + 24\,w + 8])/(2w + (1-i))$

The previous exercise raises one last question. Can the streamlines in the z-plane be produced from the imaginary part of $\Omega^{-1}(w)$? Figure 6 represents our attempt to plot the inverted dipole using $\mathrm{Im}(\Omega^{-1}(w))$ in a contour plot function. The result is not a good reproduction of figure 3 or of Needham's [20], but it does suggest that further experimentation with constants and continuations would eventually reach that goal. If one accepts this, then it validates the direction arrows of $\Omega$ and $\Omega^{-1}$ in [15] on p. 530.