Notes on VII. Dirichlet's Problem

P. 555 [exercise]

T(a) = $\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{R^2-|a|^2}{|z-a|^2}\right]T(\theta)\,d\theta$

We need only show that

$$|z-a|^2 = R^2 + r^2 - 2Rr\cos(\theta-\alpha)$$

where z = $Re^{i\theta}$ and a = $re^{i\alpha}$. Since the derivation is a simple matter of substitution into the cosine formula $\left(c^2 = a^2 + b^2 - 2ab\cos(A)\right)$, where A is the angle between z and a (not the same a as in the cosine formula), it's hard to see why it justifies an exercise.

P. 557 *We try to follow the logic of the derivation of a conformal mapping of the disc such that 0 is sent to a.*

We assume there is an M_a(z) that interchanges 0 and a about the midpoint of the straight line segment 0a. (Should 0a be a segment of an h-line perpendicular to L?)

z* = $M_a(z)$
0* = a
a* = 0

Figure 33[a] shows three circular arcs, the chord B, the h-line L, and the arc A, all passing from z to z*. The author says that these are equidistant curves, but I think we we must prove this. I do not find this configuration presented earlier in the text. We stipulate that the h-line L intersects a0 at the midpoint m, while A passes through 0. Then am equals m0. We use trigonometry to assure that the two angles signified by dots at z are equal to the two angles signified by dots at z*.

*The black angle tz0 and the shaded angle z*z0 are equal. The proof is immediate from the figure and is left to the reader.*

zt and 0t are intersecting tangents, so their tangent angles are equal and the triangle 0tz with vertex at t has equal base angles. Tangent 0t is parallel to chord B because B is the base of a triangle with equal sides and a vertex at 0, which is the point of tangency of 0t. Both B and 0t are orthogonal to the orthogonal bisector of B. Therefore the black and shaded angles at z are equal. Since the figure is symmetrical about the orthogonal bisector of B, the two dots at z signify angles that are equal to the black and shaded angles.

Now the image A* = B of A must be the same h-distance from L as A is and it must contain a, because 0* = a. We know that B is the right distance because the angles marked by the dots are equal at z and equal at z* and due to the conformality of M_a, they remain equal under the movement. Therefore a must fall on B.

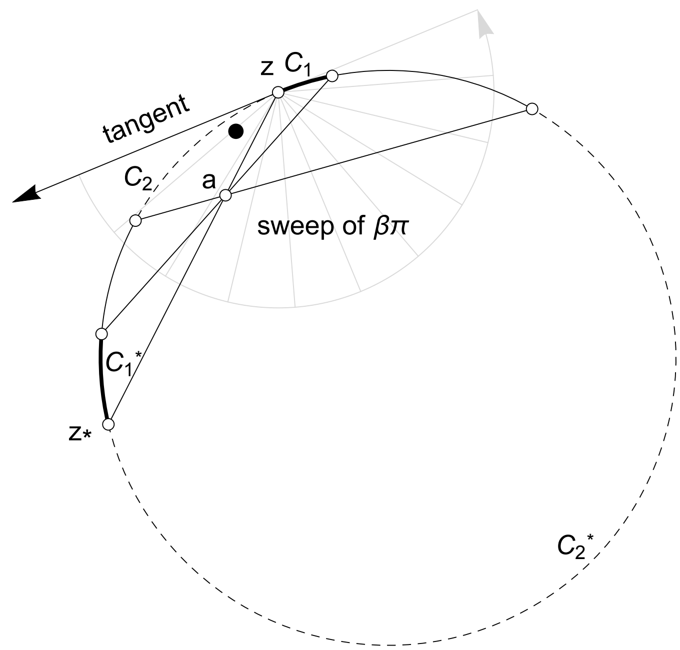

Figure 1. Dirichlet's problem as a approaches z

P. 558, Subsection 3, paragraph 3, line 4 [exercise]

If a arrives at z while travelling in a direction making an angle $\beta\pi$ with C, then [exercise] T(a) approaches [$\beta T_1 + (1 - \beta) T_2$].

We can work out some of the logic by comparing the equation with Figure 34[a]. The angle $\beta\pi$ indicated by the black dot lies between a tangent at z and the line segment za on zz*. Since $\beta\pi$ is an angle, β must be a real fraction. If we allow $\beta\pi$ to begin at the tangent at z and sweep anti-clockwise through C (Figure 1), one can see that a complete sweep completes the angle π, so we let $0 \le \beta \le 1$. This is also suggested by the factor $(1 - \beta)$ in the equation.

As **a** approaches z, its temperature becomes more strongly influenced by the temperatures on the arcs C_1 and C_2. The temperature at **a** is a sum of the temperature potentials contributed by these two arcs. The proportion contributed by each depends on the length of the arcs C_1^* and C_2^*.

Let T_1 and T_2 be the boundary temperatures on the arcs C_1 and C_2, respectively. We examine the end points as β increases from 0 to 1. At $\beta = 0, \beta\pi = 0$, so the whole of C is allocated to C_2^*. Then T(a) approaches $T_2 = [0 \cdot T_1 + (1 - 0) T_2]$ as a approaches z. When β reaches $1, \beta\pi = \pi$, the whole of C is allocated to C_1^*, so T(a) approaches $T_1 = [1 \cdot T_1 + (1 - 1) T_2]$ as a approaches z. Hence, we can say that T(a) approaches $[\beta T_1 + (1 - \beta) T_2]$ if **a** arrives at z while travelling in a direction making an angle $\beta\pi$ with C.

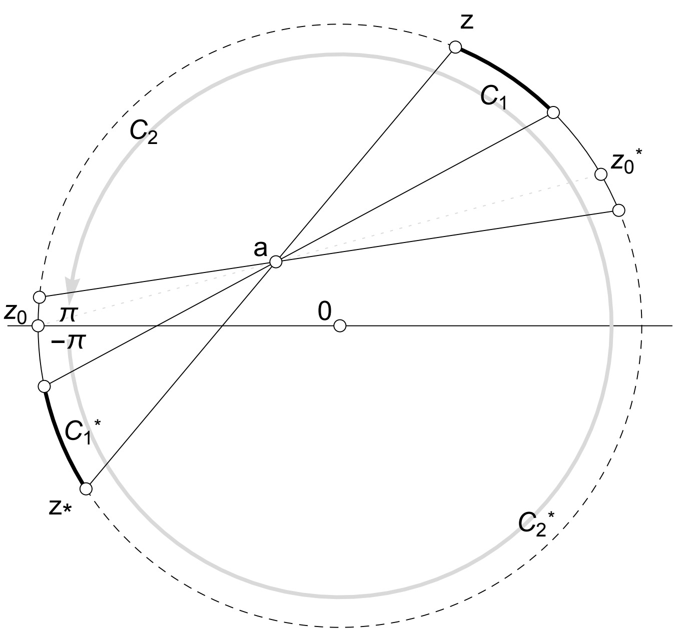

Figure 1. Dirichlet's problem as a approaches z

P. 558. *(30) may be re-expressed [why?] as*

$$T(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} T(\theta)\, d\theta^*$$

See Figure 1 b.* Since $M_a(z)$ swaps 0 and a, T(a) is equivalent to $T(0^*)$ in (30). The temperatures of points on the boundary are constant and for the sake of an example, we let T_1 be constant on C_1, T_2 be constant on C_2, and 0 on all other points. $z^* = M_a(z)$ and $\theta^* = \arg(z^*)$.

In (30), integrating $T(\theta^*)$ dθ from -π to π in the anticlockwise direction means beginning with the temperature $T_0 = T[(-\pi)^*] = T(\arg(M_a(-r)))$ and summing all the temperature points $T(\theta^*)$ over the angle 2π. This sum is then divided by 2π to obtain the average $T(\theta^*)$.

In the new expression, we integrate T(θ) over dθ^*. Hence, we begin by finding the temperature $T_0 = T(\theta_0) = T[(-\pi)^*] = T(\arg(M_a(-r)))$ [because even with dθ^*, the integration is from -π to π], and we again sum all the temperature points T(θ) over the angle 2π and divide the integral by 2π to obtain the average.

Since T(a) = $T(0^*)$, and the same temperature points are integrated in both (30) and the re-expression, the two integrals are equal, the two equations are equivalent, and the re-expression is valid.

This result is a bit startling, because the re-expression does not explicitly account for |z-a|.

*Figure 1b does not depend on figure 1. It was inserted after finishing figures 2-4.

P. 559. Schwarz's Deduction

$$\frac{R^2-r^2}{R^2+r^2-2\,\mathrm{Rr}\cos(\theta-\alpha)} = \frac{R^2-r^2}{|z-a|^2} = \frac{(R+r)\,(R-r)}{\rho^2}$$

$$= \frac{\rho\sigma}{\sigma^2} = \frac{\rho}{\sigma}$$

Hence

$$\mathrm{T(a)} = \frac{1}{2\,\pi}\int_{-\pi}^{\pi}\frac{\rho}{\sigma}\,T(\theta)\,d\theta$$

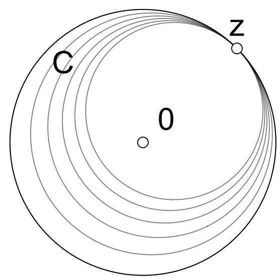

Fig. 2. Horocycles of Poisson's Kernel, z = Exp(iπ/4)

$\mathcal{P}_a = (R^2 - r^2)/|z-r|^2$

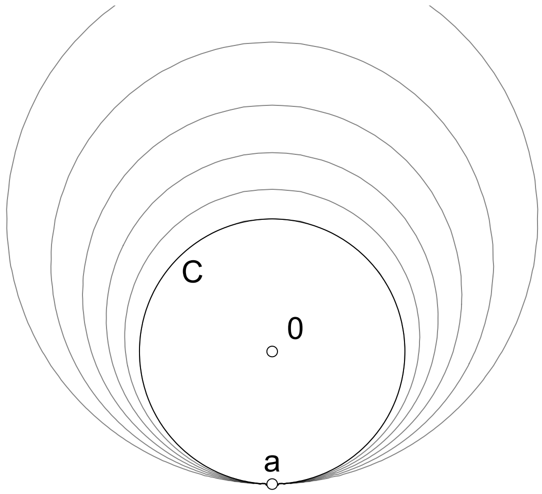

Fig. 3. Horocycles of Poisson's Kernel, a = –i

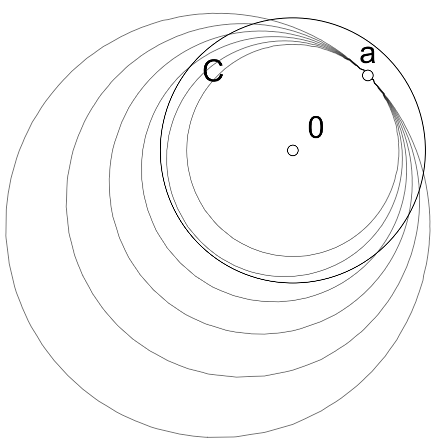

Fig. 4. Horocycles of Poisson's Kernel, a = .8 Exp(iπ/4)

P. 559. *...(with z fixed) the level curves of $\mathcal{P}_a$ are the circles which are tangent to C at z (i.e. horocycles), with $P_a = 0$ being C itself.*

The statement stipulates a fixed z, presumably located on C (Figure 2). The fact that $\mathcal{P}_a = 0$ simply means that **a** also lies on C, so that $R^2 - r^2 = 0$. When **a** is allowed to vary on C, it describes a circle. The horocycles are plotted with the ContourPlot function.

I found that $\mathcal{P}_a$ has level curves which are horocycles whether one fixes z or a. In Fig. 3, it is **a** that is fixed at -i. This means that **a** is on the boundary, so $\mathcal{P}_a$ is always 0 for |z| = 1, but other horocycles appear for higher |z|. But what if **a** is a fixed point inside C as in Fig. 4? Here we can say: The level curves of $\mathcal{P}_a$ are tangent to a circle of radius |**a**| at **a** and they are symmetrical on **a**0. This would also describe Fig. 3.