Notes on Chapter 12, §VI, §§4, Interior Mappings, Vortices, Sources, §§5, An Example: Automorphisms of the Disc, §§6 Green's Function.
November 23, 2019, 12:10 pm PST

P. 546, ¶ 3.

Insert a vortex into B at an arbitrary interior point p, ... See [27].

[27] shows non-circular but closed streamlines and curved lines of potential which appear to be curved rays. Is there a procedure for plotting this figure, or is it just the object of a thought experiment?

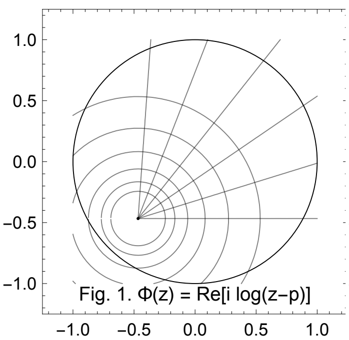


Fig. 1. Φ(z) = Re[i log(z−p)]

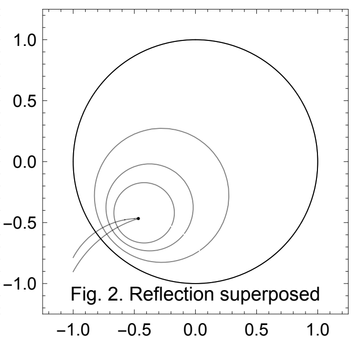


Fig. 2. Reflection superposed

P. 549. §VI, §§5 An Example

We notice that in figures [25]-[28] the only specific example of a transformation f(z) to this point is B = C (the boundary of a region---also a unit disc in this case---equals the boundary of the unit disc). Figure [29] illustrates a mapping between two vortices on the unit disc. [29b] must be the vortex with S = -2π:

$$\overline{\tilde{H}} = - \frac{i}{\overline{w}}$$

So

$$\tilde{H} = \tilde{\Omega}'(w) = \frac{i}{w}$$

$$\tilde{\Omega}(w) = i \operatorname{Log} w$$

Figure [29a] illustrates a vortex centred at p, so

$$\overline{H} = \frac{-i}{\overline{z}-\overline{p}}$$

$$H = \frac{i}{z-p}$$

$$\Omega(z) = i \log(z-p)$$

= i (ln|z-p| + i arg(z-p));

= -arg(z-p) + i ln|z-p|

Φ(z) = -arg(z-p)

Ψ(z) = i ln|z-p|

If we plot this vortex we will not get an image that looks like [29a]. But in order to complete the image by the method of [reflected] images *we superpose the real vortex at p with its reflection in C*.... Here is an opportunity for confusion, because one might think that C applied only to the disc on the right [29a] and that the disc on the left [29b] (the one in which we are reflecting the "real" vortex) would be named B. But Needham stipulated that B = C, so we take Figure [30] as an indication that we are actually reflecting an image (not shown in [29]) in the disc of [29a]. Figure 1 is a plot of a few potentials and streamlines of Ω(z) = i log(z-p). The streamlines are concentric and the potentials are rays. Figure 2 reflects Ω(z) across the unit circle containing p and superposes the reflection on the original. The equation is (19), with $\delta = 0$.

$$\Omega(z) = i \log\left[\frac{z-p}{\overline{p}\,z-1}\right] - \delta \qquad (19)$$

The streamlines in Figure 2 are not concentric circles and the potentials are curved line segments.

p. 549, last paragraph, [exercise]

In [29b] we chose to measure circulation and flux from a boundary point, and we now choose to do the same in [29a]. In terms of the above equation this is equivalent [exercise] to demanding that the constant δ be an arbitrary real number.

Since we choose a = $\tilde{\Omega}(\tilde{a})$, then Ω(z) = $\tilde{\Omega}$(w) = i log w,

$$\Omega(z) = i \log w = i \log\left[\frac{z-p}{\overline{p}\,z-1}\right] - \delta$$

$$\log w = \log\left[\frac{z-p}{\overline{p}\,z-1}\right] + i\,\delta$$

$$e^{\log w} = e^{i\delta}\, e^{\log\left[\frac{z-p}{\overline{p}\,z-1}\right]}$$

$$w = e^{i\delta}\left[\frac{z-p}{\overline{p}\,z-1}\right] \qquad (1)$$

Also, since f(z) = w, f(p) = 0. Since the mapping from z to w is fully determined by three real numbers, and since two of them have already been set by the choice of p, the remaining real number must be δ. This could have been deduced from (19), but I think it is easier to see in (1) which shows that δ is the angle by which the vortex in [29b] is rotated.

Vasco has offered a solution based on calculation rather than solely on the premises of the construction. I try to reproduce it here.

$\Omega(\tilde{a}) = 0$, so

$$\delta = \mathrm{i}\,\log\left[\frac{a-p}{\overline{p}\,a-1}\right]$$

$$\overline{p}\,a - 1 = \overline{a}(\overline{p}\,a - 1)/\overline{a} = (\overline{p}\text{-}\overline{a})/\overline{a}$$

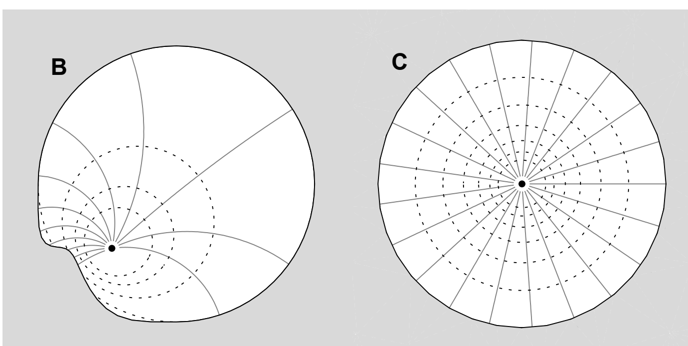
$$\delta = \mathrm{i}\,\log\left[\frac{a-p}{(p-a)/\overline{a}}\right] = \mathrm{i}\,\log\left[-\frac{\overline{a}(a-p)}{(a-p)}\right]$$

Let $q = -\frac{\overline{a}(a-p)}{(a-p)}$.

Since a is on the boundary, $|\overline{a}| = 1$, and since the quotient of the absolute values of conjugates equals 1, $|q| = 1$. Then we can write $q = e^{\mathrm{i}\phi}$, ϕ a real angle.

$$\delta = \mathrm{i}\,\log\left[e^{\mathrm{i}\phi}\right] \;=\; -\phi$$

Since ϕ is a real number based on choice of $\overline{a}$, which is based on arbitrary choice of a in [29b], then δ = -ϕ must be an arbitrary real number.



P. 550 Green's Function

Figure 3 is a facsimile of [28] but missing the arrowheads and with only approximate shape of the boundary B. Unlike [28], the potential lines of figure 3 intersect the boundary, showing that this figure was not created by reflection in B. The underlying image was simply masked out from B outward. But at least the streamlines intersect at right angles, for the most part. Getting this right is an ongoing project.

p. 551 Derivation of $\mathcal{G}_p$ (Green's formula for temperature from vortex in (19).

$$\Omega(z) = \mathrm{i}\,\log\left[\frac{z-p}{\overline{p}\,z-1}\right] - \delta\text{, where } \delta \text{ is constant (19)}$$

From p. 549 [exercise], above, we know that δ is a real number. The source is the dual of the vortex: $\hat{\Omega}(z) = -i\Omega(z)$.

$$\hat{\Omega}(z) = -i\Omega(z) = \log\left[\frac{z-p}{\overline{p}\,z-1}\right] + i\delta$$

$$\hat{\Phi}(z) = \text{Re}(\log\left[\frac{z-p}{\overline{p}\,z-1}\right] + i\delta) = \ln\left|\frac{z-p}{\overline{p}\,z-1}\right|$$

$$\mathcal{G}(z) = -\hat{\phi}(z) = -\ln\left|\frac{z-p}{\overline{p}\,z-1}\right| = -\ln|f(z)|, \text{ where } f(p) = 0$$

Needham immediately rewrites $\hat{\Omega}(z)$ as $\Omega(z)$ and $\hat{\Phi}(z)$ as $\Phi(z)$.

$$\mathcal{G}(0) = -\ln\left|\frac{-\rho}{-1}\right| = -\ln|p|$$

$$\mathcal{G}(p) = -\ln|0| = \infty$$

$\mathcal{G}(p)$ has a pole p and a harmonic dual $\mathcal{H}(z)$.

$\Omega(z) \approx \log(z-p)$ near p. Thus if $z = p + \rho e^{i\phi}$, then

$$\mathcal{G}(p) = -\ln\rho$$

$$\mathcal{G}_p(z) = -\Phi = -\text{Re}[\log(z-p)] = -\ln\rho$$

P. 552. *Returning to (22), the precise version is that $\mathcal{G}_p$ differs from -ln ρ by a function $g_p(z)$ which is harmonic throughout R:*

$$\mathcal{G}_p(z) = -\ln\rho + g_p(z)$$

What is $\mathcal{G}_p(z)$ and how do we know it is harmonic? We discard the hat for the source $\Omega(z)$ and begin with $\rho = |z-p|$.

$$\Omega(z) = \log[\frac{z-p}{\overline{p}\,z-1}] + i\delta$$

$$= \ln\frac{\rho}{|\overline{p}\,z-1|} + i\arg(\frac{z-p}{\overline{p}\,z-1}) + i\delta$$

$$\mathcal{G}_p(z) = -\Phi = -\text{Re}[\ln\frac{\rho}{|\overline{p}\,z-1|} + i\arg(\frac{z-p}{\overline{p}\,z-1}) + i\delta]$$

$$= -\ln\frac{\rho}{|\overline{p}\,z-1|} = -\ln\rho + \ln|\overline{p}z-1|$$

Then $g_p(z) = \ln|\overline{p}z-1|$.

Exercise, p. 554: At the boundary point z, the local heat flux is equal to the amplification of f: $\mathcal{Q}(z) = |f'(z)|$. Using (20), the result predicts that the local heat flux at the boundary of the unit disc is given by [exercise].

$$\mathcal{Q}(z) = |f'(z)| = \frac{1-|p|^2}{|z-p|^2} = \frac{1-|p|^2}{\rho^2}$$

Let $f(z) = e^{i\delta}\left[\frac{z-p}{\overline{p}\,z-1}\right]$.

$$f'(z) = e^{i\delta}\ \frac{1\cdot(\overline{p}\,z-1)-(z-p)\,\overline{p}}{(\overline{p}\,z-1)^2} \text{ by the quotient rule}$$

$$= e^{i\delta}\ \frac{-1+|p|^2}{(\overline{p}\,z-1)^2}$$

$|f'(z)| = |e^{i\delta}\ \frac{-1+|p|^2}{(\overline{p}\,z-1)^2}| = \frac{1-|p|^2}{|(\overline{p}\,z-1)^2|}$, because $\left|e^{i\delta}\right| = 1$, and $|p| \le 1$.

$|(\overline{p}\,z-1)^2| = |(\overline{p}\,z\ -\ 1)|^2 = (\overline{p}z-1)\,(p\overline{z}-1)$

$= |p|^2\,|z|^2 - \overline{p}z - p\overline{z} + 1$

At the boundary of the unit disc, $|z| = 1$, so

$|(\overline{p}\,z-1)^2| = |p|^2 - \overline{p}z - p\overline{z} + 1$

$|z-p|^2 = (z-p)\,(\overline{z}-\overline{p}) = |z|^2 - \overline{p}z - p\overline{z} + |p|^2$

$= |p|^2 - \overline{p}z - p\overline{z} + 1$

Hence $|(\overline{p}\,z-1)^2| = |z-p|^2$, so if $\rho = |z-p|$, $|f'(z)| = \frac{1-|p|^2}{\rho^2}$, which was to be shown.

*Of course Q(z) may instead be calculated directly by substituting (21) into (24), but this is a little easier said than done **[exercise]**.*

See Exercise 19.