Notes on Chapter 12, Section VI, §§1-3, The Physics of Riemann's Mapping, pp. 540-544
May 13, 2019, 3:10 pm PST

P. 542, ¶ 1. *The geometric signature of the critical point is the angle ($\pi/4$) between the streamline and the equipotential.*

This looks right on inspection, but how does one know? There is a visual way to deduce that the angle ($\pi/4$) between the streamline and the equipotential is $\pi/4$. The answer can be found in [15], p. 530. Figure [25b] corresponds to $\Omega^{-1}(w)$ in [15] rotated by ϕ. At critical points (stagnation points), angles are doubled under $\Omega(z)$, so an angle of $\pi/2$ between a line of potential and a streamline in the w-plane ($\Omega(z)$) becomes $\pi/4$ in the z-plane ($\Omega^{-1}(w)$). The rotation of the flow does not affect the angles between potentials and streamlines.

Vasco had another suggestion: *It should be possible to show from the equation $\Phi(x,y)=2$, (the value of Φ for the equipotential through the stagnation point), that $y\prime(x=1)=1$.*

So we must first show that $\Phi(x, y) = 2$, or rather, since in this case the flow obstructed by the circle D is in the image panel of Figure [25], which is the w-plane, we must show that $\tilde{\Phi}(u, v) = 2\tilde{\boldsymbol{v}}$, where $\tilde{\boldsymbol{v}}$ is speed. From p. 540, §V, we deduce that $\tilde{\Omega}_d(w) = \tilde{\boldsymbol{v}}e^{-i\phi}\, w + \frac{\tilde{\boldsymbol{v}}\, e^{i\phi}}{w}$, where ϕ is the angle of the flow with respect to the u axis.

$$\Omega\tilde{(w)} = \tilde{\boldsymbol{v}}\, [e^{-i\phi}\, w + \frac{e^{i\phi}}{w}] = \tilde{\boldsymbol{v}}\, [\cos(-\phi) + i \sin(-\phi))\, w + \mathbf{v}\, (\cos(\phi) + i \sin(\phi))/w]$$

$$= \tilde{\boldsymbol{v}}\, [(\cos(\phi) - i \sin(\phi))\, w + (\cos(\phi) + i \sin(\phi))/w]$$

$$= \tilde{\boldsymbol{v}}\, [(\cos(\phi) - i \sin(\phi))\, (u + iv) + (\cos(\phi) + i \sin(\phi))/(u + iv)]$$

$$= \tilde{\boldsymbol{v}}\, [\cos(\phi)\, u + i \cos(\phi)\, v - i \sin(\phi)\, u + \sin(\phi)\, v + (\cos(\phi) + i \sin(\phi))\, (u - iv) / |w|^2]$$

$$= \tilde{\boldsymbol{v}}\, [\cos(\phi)\, u + \sin(\phi)\, v + i(\cos(\phi)\, v - \sin(\phi)\, u) + (\cos(\phi)u - i \cos(\phi)v + i \sin(\phi)u + \sin(\phi)\, v) / |w|^2]$$

$$= \tilde{\boldsymbol{v}}\, [(\cos(\phi)\, u + \sin(\phi)\, v + (\cos(\phi)u + \sin(\phi)\, v)/ |w|^2 + i[(\cos(\phi)\, v - \sin(\phi)\, u) + (-\cos(\phi)v + \sin(\phi)\, u) / |w^2|)]$$

$$\tilde{\phi} = \mathrm{Re}(\Omega\tilde{(w)}) = \tilde{\boldsymbol{v}}\, [(\cos(\phi)\, u + \sin(\phi)\, v + (\cos(\phi)u + \sin(\phi)\, v) / |w^2|]$$

At the stagnation point $\tilde{\alpha}$, $u = \cos(\phi + \pi) = -\cos(\phi)$, $v = \sin(\phi + \pi) = -\sin(\phi)$, and $|w^2| = 1$.

$$\tilde{\phi}_{\tilde{a}} = \tilde{\boldsymbol{v}}\, [(\cos(\phi)\, (-\cos(\phi)) + \sin(\phi)\, (-\sin(\phi)) + (\cos(\phi)(-\cos(\phi)) + \sin(\phi)\, (-\sin(\phi))/1]$$

$$= -2\tilde{\boldsymbol{v}}\, (\cos^2 \phi + \sin^2\phi) = -2\tilde{\boldsymbol{v}}$$

$$\tilde{\phi}_{\overline{b}} = \tilde{\boldsymbol{v}}\ [(\cos(\phi)\cos(\phi) + \sin(\phi)\sin(\phi) + (\cos(\phi)\cos(\phi) + \sin(\phi)\sin(\phi))/1]$$

$$= 2\tilde{\boldsymbol{v}}\ (\cos^2\phi + \sin^2\phi) = 2\tilde{\boldsymbol{v}}$$

Regardless of the slopes of the streamlines and potentials, the difference $[\tilde{\Phi}] = 2\tilde{\boldsymbol{v}} - (-2)\tilde{\boldsymbol{v}} = 4\tilde{\boldsymbol{v}}$.

Finding the slope dv/du

To find *dv/du*, the slope of the potential curve, we let $\phi = 0$, because we know that ϕ rotates the potential and flux curves by the same amount. Given that we are working in the w-plane, we are seeking v', i.e. dv/du for $\tilde{\Phi} = \mathrm{Re}[\tilde{\Omega}]$.

$$\tilde{\phi} = \mathrm{Re}(\Omega(\tilde{w})) = \tilde{\boldsymbol{v}}\ [(\cos(\phi)\, u + \sin(\phi)\, v + (\cos(\phi)\, u + \sin(\phi)\, v) / \left| w^2 \right|] = -2\tilde{\boldsymbol{v}}$$

$$\Rightarrow\ u + u \big/ \left(u^2 + v^2\right) = -2$$

$$u\left(u^2 + v^2\right) + u = -2\left(u^2 + v^2\right)$$

$$u^3 + uv^2 + u = -2u^2 - 2v^2$$

$$v^2(u+2) = -\left(u^3 + 2u^2 + u\right)$$

$$v^2 = \frac{-\left(u^3 + 2u^2 + u\right)}{u+2} \qquad (1)$$

$$2v\,dv = -\frac{2\left(u^3 + 4u^2 + 4u + 1\right)}{(u+2)^2}\, du$$

$$\frac{dv}{du} = -\frac{1}{v}\ \frac{\left(u^3 + 4u^2 + 4u + 1\right)}{(u+2)^2}$$

But we know that v = 0 at $\tilde{\alpha}$, so $\frac{dv}{du}$ is undefined and cannot be used to find the angle between the potential and the streamline at the stagnation point. But we still have an answer in paragraph 1, above.

Vasco suggested we let $u = -1 + \epsilon$ and find the limit as ϵ goes to 0. Infinitesimals at higher powers vanish. We are left with ϵ in the numerator and v = 0 in the denominator, so this approach also leaves the derivative undefined unless I am not applying the suggestion correctly or my calculation of dv/du is wrong.

Let $u = -(1 + \epsilon)$ because in the UHP, u approaches -1 from the left.

$$\frac{\mathrm{dv}}{\mathrm{du}} = \mathrm{Lim}_{\epsilon\to 0}\left[-\frac{1}{v}\;\frac{\left(-(1+\epsilon)^3+4\,(-(1+\epsilon))^2+4\,(-(1+\epsilon))+1\right)}{((-(1+\epsilon))+2)^2}\right]$$

$$= \mathrm{Lim}_{\epsilon\to 0}\left[-\frac{1}{v}\;\frac{\epsilon\left(\epsilon^2-\epsilon-1\right)}{(\epsilon-1)^2}\right] = \frac{\epsilon}{0\cdot 1} = \frac{0}{0}$$

This is still undefined, but why not also let v = $(0 + \epsilon)$ very near to the stagnation point? We find that the limit of $\frac{\epsilon}{\epsilon}$ is still undefined. But the very fact that the ratio very near the limit is 1 suggests that the slope of the infinitesimal segment is 1 and the angle of approach is $\pi/4$.

Another attempt

Take the square root of both sides of (1) and then find dv/du:

$$\mathrm{v} = \sqrt{-\frac{u\,(u+1)^2}{u+2}}$$

$$\mathrm{dv} = -\frac{(\mathrm{u}+1)\;\left(\mathrm{u}^2+3\;\mathrm{u}+1\right)}{\sqrt{-\frac{\mathrm{u}\;(\mathrm{u}+1)^2}{\mathrm{u}+2}}\;\;(\mathrm{u}+2)^2}\;\mathrm{du} \qquad (2)$$

Let u = -1 - ϵ, and disregard all powers of ϵ greater than 1 in expansions.

$$\frac{\mathrm{dv}}{\mathrm{du}} = \mathrm{Lim}_{\epsilon\to 0}\left[\frac{\left((-\epsilon-1)^2+3\,(-\epsilon-1)+1\right)\epsilon}{(1-\epsilon)^2\sqrt{-\frac{(-\epsilon-1)\,\epsilon^2}{1-\epsilon}}}\right] = \mathrm{Lim}_{\epsilon\to 0}\left[\frac{(1-3+1)\,\epsilon}{1\cdot\sqrt{-\frac{-\epsilon^2}{1}}}\right] = \mathrm{Lim}_{\epsilon\to 0}\left[\frac{-\epsilon}{\epsilon}\right]$$

Taken to $\epsilon = 0$, this results in another $\frac{0}{0}$, but it might also be interpreted as a slope of -1 very near the stagnation point. This seems a more trustworthy approach than the previous which depended on setting v to ϵ. It now appears that one can say that very near $\tilde{\alpha}$, the potential curve on the upper left makes an angle of $\pi/4$ with the streamline.

Vasco suggested finding y' (v' here) prior to substituting for Φ. Then substitute ±2 for Φ and substitute numbers very close to ±1, such as .9999 and 1.0001, for 1-ϵ and 1+ϵ. This results in values very close to 1 or -1, showing that y' is ±1 when x is ±1. Similarly, if one substitutes .9999 and 1.0001 for 1-ϵ and 1+ϵ in (2), above, the results are -1.00015 and 1.00015 for the positive and negative square root. The advantage of postponing the substitution of Φ until after the calculation of y' or v' is that one can test several substitutions systematically without recalculating the derivative and one sees numerical results that approximate the slope of ±1.

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Needham concludes *Since we know that [$\tilde{\Phi}$] = 4v, we deduce that...*

> *A one-to-one, conformal mapping between the two grids is obtained if and only if the disc is*

inserted into a flow of speed $v = [\Phi]/4$.

Since $[\tilde{\Phi}] = 4\tilde{\boldsymbol{v}}$, and we have set $[\tilde{\Phi}] = [\Phi]$, then $\tilde{\boldsymbol{v}} = [\tilde{\Phi}]/4 = [\Phi]/4$.