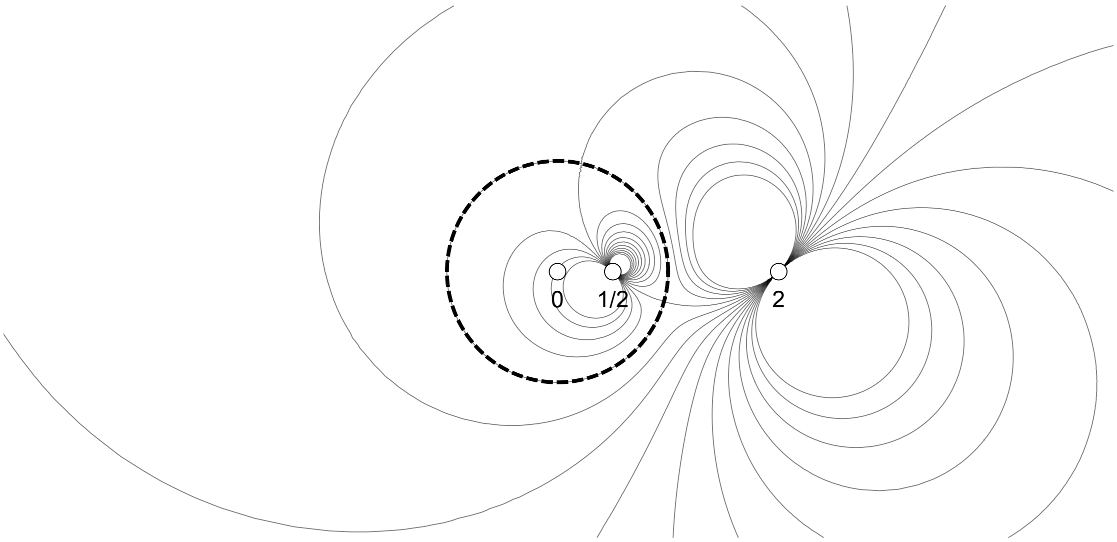
Notes on Chapter 12, Section V, Flow Around an Obstacle, Schwartz symmetry on dipole
April 19, 2019, 1:40 pm PST

Figure 10. Ω_u = (1+i)/(z−2) + (1−i) z/(1−2z)



P. 535. (13) Figure 10 (facsimile of [20])

On p. 532, Needham writes of *the flow of a dipole located at 2, having a dipole moment -(1+i), and thus represented by the complex potential:*

$$\Omega_u(z) = \frac{(1+i)}{z-2}. \qquad (13)$$

This only works if "the flow of a dipole" is defined as

$$H_u(z) = (\Omega_u(z))' = \frac{-(1+i)}{(z-2)^2}$$

Then

$$\Omega_u(z) = \int H_u(z)\, dz = \frac{(1+i)}{z-2}$$

$$\Omega_u^\dagger(z) = \Omega_u\left(\frac{1}{z}\right) = \frac{(1+i)}{\left(\frac{1}{z}\right)-2} = \frac{(1-i)\,z}{1-2\,z} \qquad \text{(inversion in the unit circle)}$$

$$\Omega_d(z) = \Omega_u(z) + \Omega_u^\dagger(z) = \frac{1+i}{z-2} + \frac{(1-i)\,z}{1-2\,z}$$

$$= \frac{i\,(x-y-2)+x+y-2}{x^2-4\,x+y^2+4} + \frac{i\,(x+y)+x-y}{4\,x^2-4\,x+4\,y^2+1}$$

$$\phi_d = \text{Re}[\Omega_d] = \frac{x+y-2}{x^2-4\,x+y^2+4} + \frac{x-y}{4\,x^2-4\,x+4\,y^2+1}$$

$$\psi_d = \text{Im}[\Omega_d] = \frac{x-y-2}{x^2-4\,x+y^2+4} + \frac{x+y}{4\,x^2-4\,x+4\,y^2+1}$$

To program the plot, it works just as well to use $\phi_d = \text{Re}[\Omega_d]$ and $\psi_d = \text{Im}[\Omega_d]$.

P. 536, ¶ 2. *That this* [$\frac{(1-i)\,z}{1-2\,z}$] *is indeed a dipole at (1/2) may be seen by rewriting it as....*"

$$\Omega_u^\dagger(\text{z}) = \frac{(1-i)\,z}{1-2\,z} = (1\text{-i})\ \frac{1}{2}\ \frac{2\,z}{2\,z-1} = (1\text{-i})\ \frac{1}{2}\ \frac{1+2\,z-1}{2\,z-1}$$

$$= (1\text{-i})\ \frac{1}{2}\ (\frac{1}{2\,z-1} + \frac{2\,z-1}{2\,z-1}) = (1\text{-i})\ \frac{1}{2}\ (\frac{1}{2\,z-1} + 1)$$

$$= \text{-}(1\text{-i})\ (\frac{1}{4}\left[\frac{1}{z-1/2}\right]\text{-}\ \frac{1}{2}) = -\frac{(1-i)}{4}\left[\frac{1}{z-1/2}\right] - \frac{(1-i)}{2}$$

Out[65]=

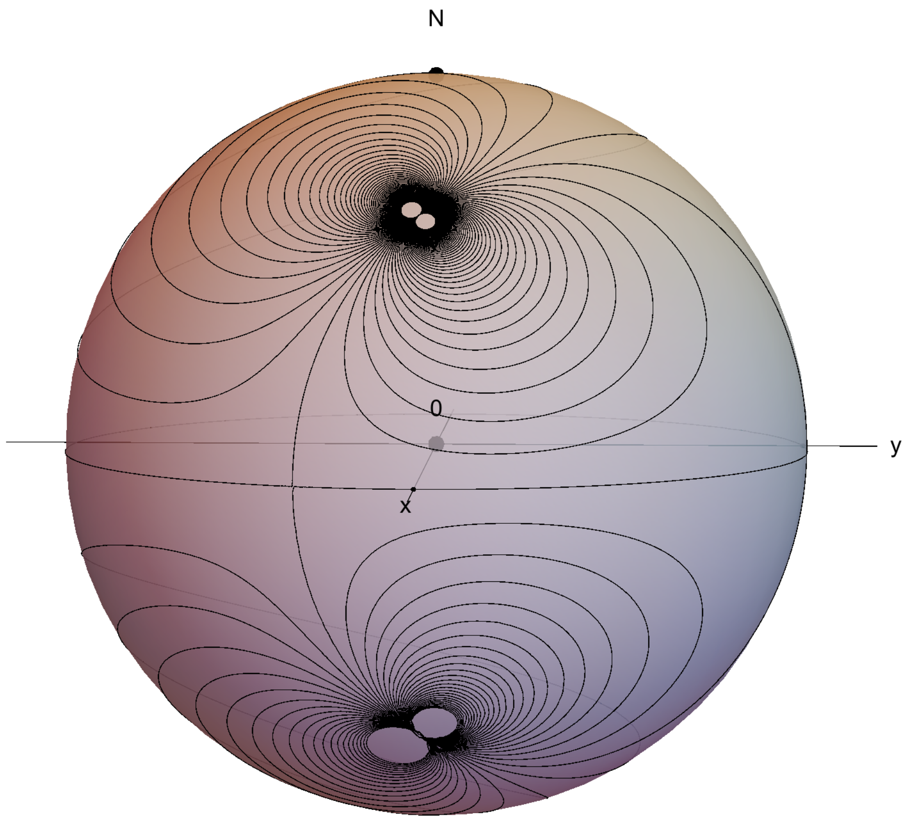


Figure 11a. Reflected dipoles
projected onto Riemann sphere

Out[70]=

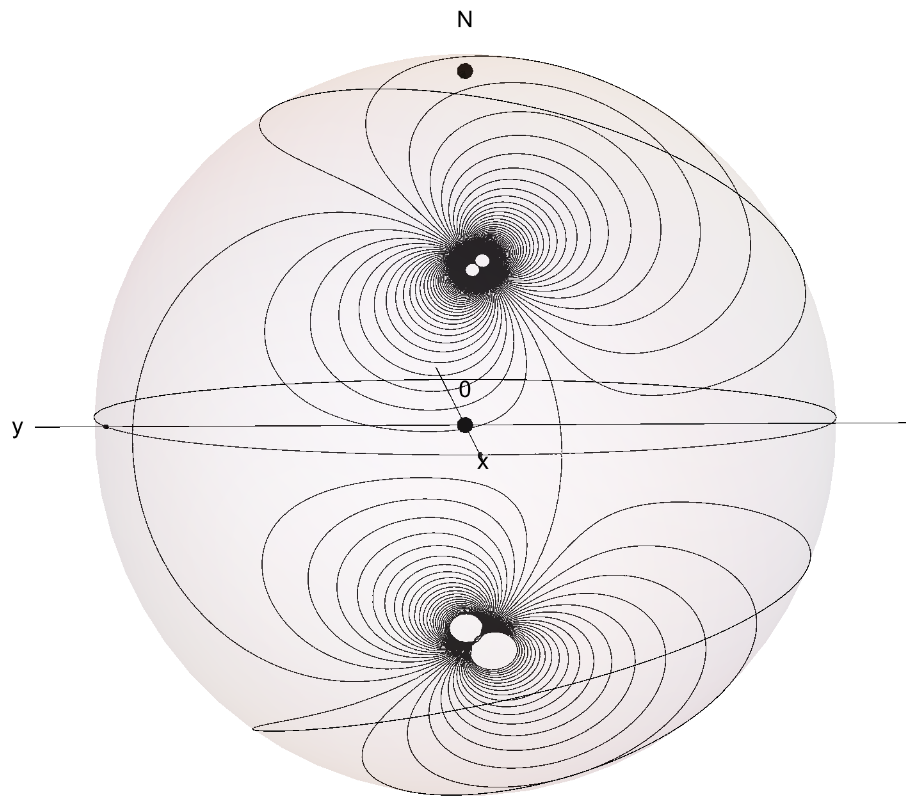


Figure 11b. View of 11a from transparent back side

P. 536, and Figure [21], p. 537. *Behold figure [21]! Note that on the sphere the strengths of the two dipoles become equal.*

Figure 11a replicates the essential features of [21]: the reflection across the unit circle in the ℂ plane, which becomes the equatorial circle on the Riemann sphere, the preservation of symmetry, and the equalization of strengths of the dipoles. The points that were projected were scavenged from the list of points created by ContourPlot[] in *Mathematica*, which uses them to plot the streamlines in 2D graphics. We used the equations on page 146 to project the 2D points onto the 3D sphere. The view from the back side is the same as what one might see from the inside (Figure 11b).