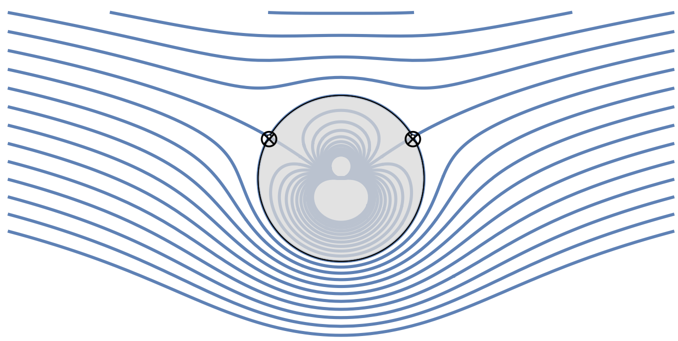
Notes on Chapter 12, Section V, Flow Around an Obstacle, 2 An Example.
April 2, 2019, 6:45 pm PST

Figure 1. Contour plot of stream function ψ for uniform flow on doublet and vortex

Out[1202]=



P. 529, ¶ 2, ln -1. *The reader is strongly encouraged to use a computer to verify this figure*, i.e. [13].

Based on the discussion on p. 488, ¶2, §§6, the vortex has an equation $\overline{H} = (S/2\pi)\,(i/(\overline{z} - \overline{p})$. Let p = 0, placing the vortex at the origin. We keep in mind that H = Ω'.

$$\Omega_{\text{dipole}} = \frac{1}{z} = \frac{x - iy}{x^2 + y^2}$$

$$H_{\text{vortex}} = -\frac{S}{2\pi}\frac{i}{z}$$

$$\Omega_{\text{vortex}} = \int H_{\text{vortex}}\,dz = -\frac{Si}{2\pi}\int\frac{1}{z}dz = -\frac{Si}{2\pi}\text{Log}(z)$$

$$\Omega = \Omega_{\text{uf}} + \Omega_{\text{dipole}} + \Omega_{\text{vortex}} = z + \frac{1}{z} - \frac{Si}{2\pi}\text{Log}(z)$$

Rewrite this as Ω = Re + i Im to find ψ as the imaginary part of Ω.

$$\Omega = x + iy + \frac{x - iy}{x^2 + y^2} - \frac{Si}{2\pi}\ln|z| + \frac{S}{2\pi}\arctan(y/x)$$

$$= \left(x + \frac{x}{x^2 + y^2} + \frac{S}{2\pi}\arctan(y/x)\right) + \left(iy - \frac{iy}{x^2 + y^2} - \frac{Si}{2\pi}\ln\left|\sqrt{x^2 + y^2}\right|\right)$$

$$= \left(x + \frac{x}{x^2 + y^2} + \frac{S}{2\pi}\arctan(y/x)\right) + i\left(y - \frac{y}{x^2 + y^2} - \frac{S}{2\pi}\ln\left|\sqrt{x^2 + y^2}\right|\right)$$

$$\psi = \text{Im}[\Omega] = y - \frac{y}{x^2 + y^2} - \frac{S}{2\pi}\ln\left|\sqrt{x^2 + y^2}\right|$$

Figure 1 is a contour plot of ψ with S = 2π. We see that the dotted lines that seem to represent the transformed flow from [12] have been pulled into the UHP as they appear in [13].

The points of stagnation are found by setting $\overline{H}$ = H = 0. Then, given our definition of H_{vortex},

$$\text{H} = 1 - \frac{1}{z^2} - \frac{S}{2\pi}\frac{i}{z} = 0 \quad \Longrightarrow \quad z^2 - \frac{S}{2\pi} i\, z - 1 = 0$$

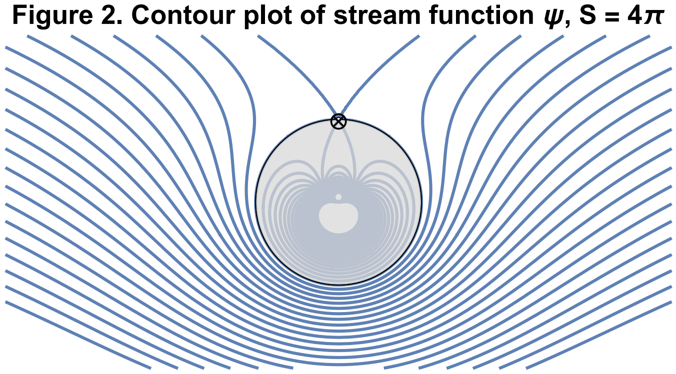
$$\text{z} = \frac{i\,\frac{S}{2\pi} \pm \sqrt{\left(\frac{-\text{Si}}{2\pi}\right)^2 + 4}}{2}$$

They plot as small white discs on the circumference of the unit circle.

In Figure 1, the dipole still appears. It could easily be masked out to make the central disc look like that in [13]. I left it in because I am curious about its status. If the dipole describes a flow within the flow, why obscure it other than to simplify the inspection of the outer flow around the unit disc? Apparently there is something to be learned from combining the dipole and vortex into a single barrier. What does the circulation of the vortex have to do with the streamlines of the uniform flow and the dipole? Vasco’s answer (p.c. Forum) is that at each point the circulation of the vortex changes the velocity of the field vectors and the direction of the streamlines. In this plot we see that the vortex does not appear as a separate entity. This is not surprising, since we are plotting flow and the vortex itself has no flux. But is does change the streamlines of the uniform flow and the dipole.

Out[1211]=

Figure 2. Contour plot of stream function ψ, S = 4π



p. 529, ¶ 2. *As you gradually increase the value of S, notice how the stagnation points on the circle move towards each other and finally coalesce at i. At what value of S does this occur?*

Since we know that z is on the y axis, the value under the square root symbol must be 0. Then

$$\text{z} = \frac{i\,\frac{S}{2\pi} \pm \sqrt{\left(\frac{-\text{Si}}{2\pi}\right)^2 + 4}}{2}$$

$$\left(\frac{-\text{Si}}{2\pi}\right)^2 + 4 = 0$$

$$-\frac{\text{S}^2}{4\pi^2} = -4$$

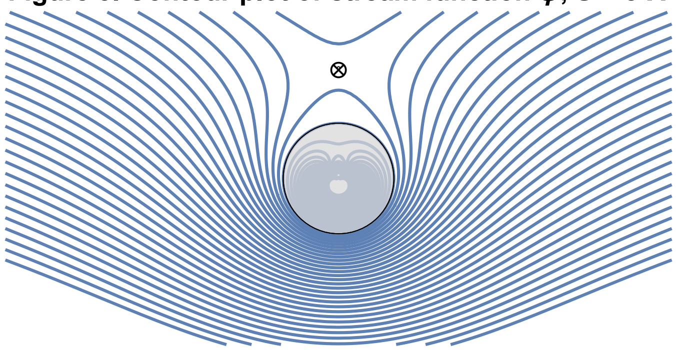
$$S^2 = 16\,\pi^2$$

$$S = 4\pi$$

Given H as I have defined it, S occurs at i when its value is 4π (Figure 2). We increase S to $5\,\pi$ (Figure 3).

Figure 3. Contour plot of stream function ψ, S = 5 π

Out[1222]=



Out[1334]=

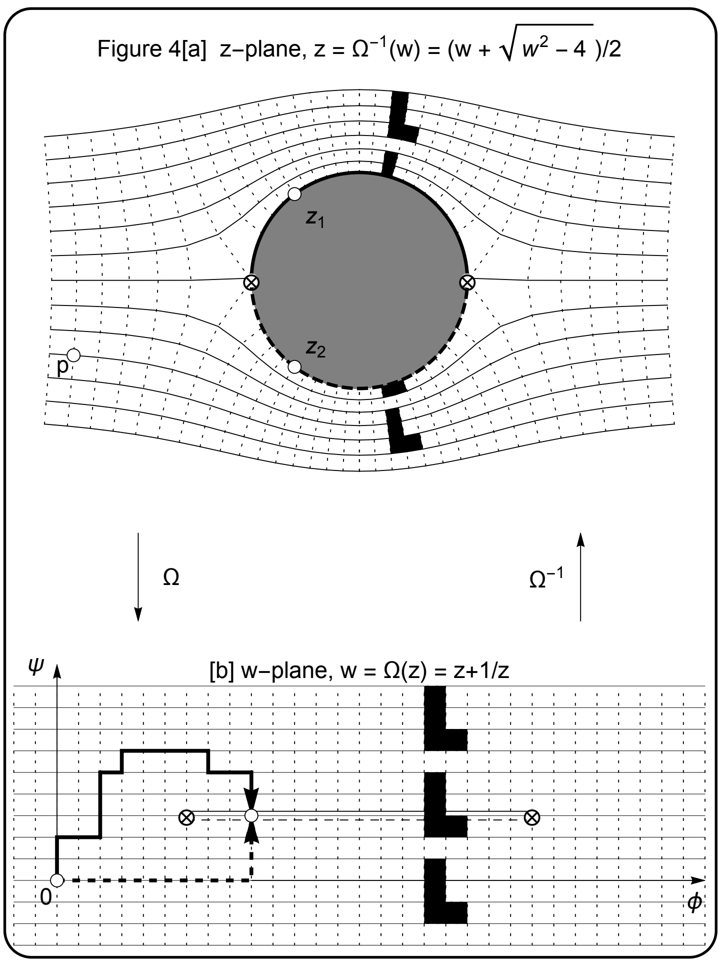


Figure 4[a] z–plane, $z = \Omega^{-1}(w) = (w + \sqrt{w^2 - 4}\,)/2$

[b] w–plane, $w = \Omega(z) = z+1/z$

Discontinuity across a cut

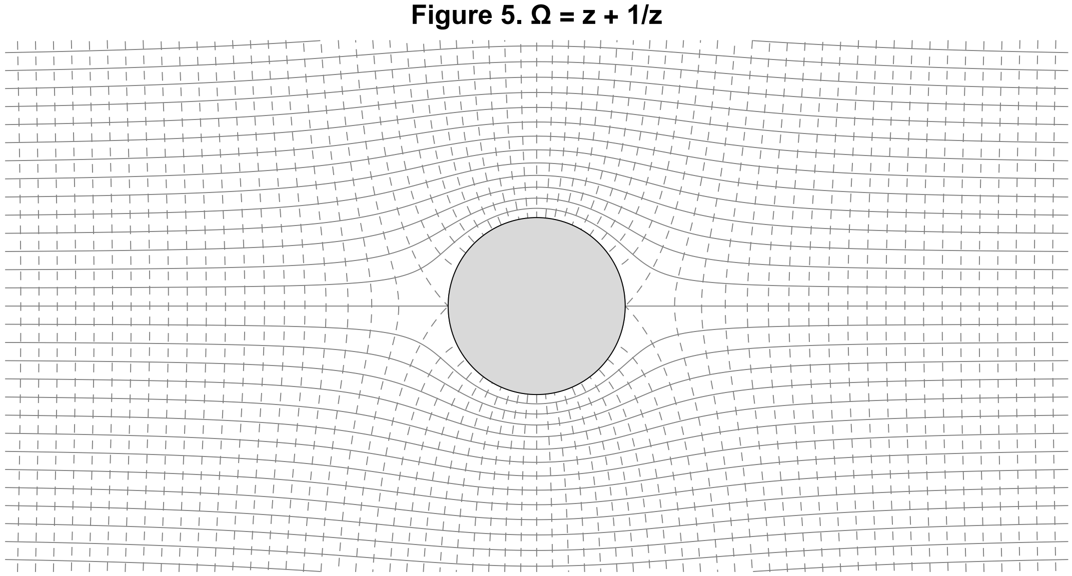
The point of this exercise is to see if the discontinuity across the cut shown in Needham's figure **[15], Ch. 12, Section V, p. 530**, can be demonstrated. Figure 4[a] was designated the "z-plane". It was created by transforming [b] with $\Omega^{-1}(w) = (w + \sqrt{w^2 - 4})/2$ applied to a grid of horizontal and vertical lines of points. The transformation replicates the lines of flow shown in [15] or Figure 5, below. It was not clear how one might go about establishing the boundary points of the polygons in [a] in order to transform [a] back into [b], so the starting point was the w-plane in [b]. The left half of [a] was created by rotating the right half about the y axis. Otherwise, the left half would be blank. The proper orientation of the split segments of the polygon in [z] was obtained by avoiding the ψ-line of zero flux, which is the circumference of the gray disc in [a] and the two cut lines in [b]. Accurate lines of flux in [a] were obtained by avoiding y = 0 and x = 0.

The discontinuity shown in [15] also appears in [b]. The orientations of the parts of the discontinuous L are correct. However, it was not achieved without some translations of the polygons in the z plane. They had to be translated by [(-15-3i) k] *before* being transformed by Ω^{-1}. This is equivalent to moving the origin in the w-plane to (15 + 3i), which puts it at the middle of the cut line. If the unit disc is removed, the middle L polygon is revealed to be stretched and distorted but continuous except for vertices.

Figure 5 shows Ω^{-1} plotted with the *Mathematica* ContourPlot[] function applied separately to ϕ and ψ. I see no essential difference in the lines of flux and potential between Figure 4a and 5, but 5 was very much easier to program.

Figure 5. Ω = z + 1/z

Out[1371]=



P. 532. To plot the dual of [12] in Figure 6, we use, from p. 509:

$$\hat{\psi} = -\phi = \mathrm{Re}[\Omega] = -\left(x + \frac{x}{x^2+y^2}\right)$$

$$\hat{\phi} = \psi = \mathrm{Im}[\Omega] = y - \frac{y}{x^2 + y^2}$$

Figure 6. Electrostatic dual $\hat{\Omega}$ of $\Omega = z + 1/z$

Out[1390]=