Notes on Chapter 12, Section V, §§4, Mapping One Flow Onto Another, Figure [24], pp. 538-540
August 23, 2019, 5 pm PST

Out[2765]=

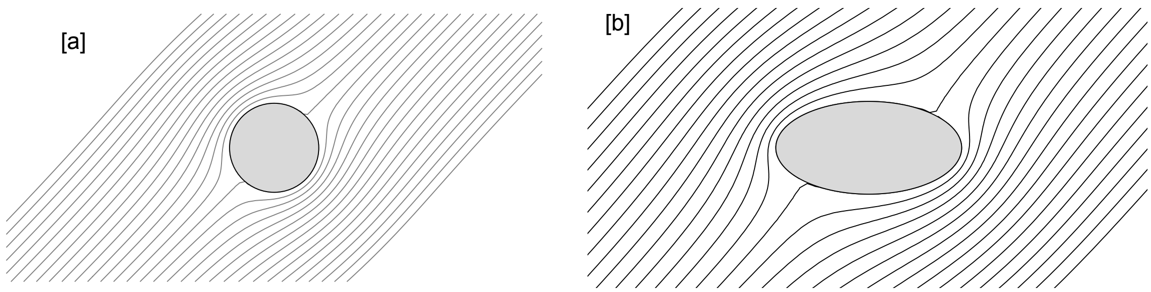

Figure 13. $\Omega_d = e^{-i\phi}z + e^{i\phi}/z$

Figure 13 shows the plot of a uniform flow at angle $\pi/4$ with a circular object inserted in the flow, as described on pp. 538-540. We harvest the points of the lines of flow (the lines of equal ψ) by means of the following function in Mathematica:

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contourLists[contourPlot_] := Cases[Normal@contourPlot, Line[list_]:→list, Infinity];
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The function creates a table of the lists of points on the level function ψ created by the function ContourPlot[] --- one list per each value of ψ. We select just those points on the lines of flux that lie outside the unit circle. We then apply our $f(z)$ to every point in every list.

f(z) = (1/2)(3z + 1/z) (18), equation for ellipse

Inspection of figure 13b shows that the lines of flow are much like those of Needham's [24b]. This looks correct, but I think it is more work than what Needham intended. What I have done is apply f to the streamlines of Ω_d, but when Needham says to map the flow round C to the flow round E, he apparently means to set $\tilde{\Omega}(w) = \Omega_d$, substitute $f^{-1}(z)$ for z, and find the stream function. See p. 540, ¶ 2. We find that this works, but it requires management of a function with multiple values of the square root. See figure 15, below.

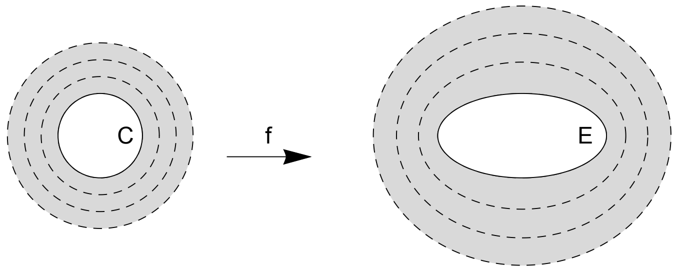

Figure 14. Facsimile of Figure [23], p. 539

Figure 14 shows that the equation of the ellipse works to transform C into E and circles around C into ellipses around E. However, we don't find that applying f to the flow round C in [24] produces a figure like that on the right side of [24]. See figure 15a, b. That is, let f(z) = (1/2)(3z + 1/z). Then applying f to $\Omega_d(z) = e^{-i\phi} z + \frac{e^{i\phi}}{z}$ and using the imaginary part in *Mathematica*'s ContourPlot[] function, does not produce the streamlines around a central ellipse as we see in [24b]. Perhaps more surprisingly, neither does it work to plot the imaginary part of $\tilde{\Omega}(w) = f^{-1}(w) + \left[1 \big/ f^{-1}(w)\right]$ in ContourPlot[]. One might expect it to work, because of the statements in ¶ 2, p. 540: "*Choose any analytic mapping [such as f(z) = (1/2)(3z + 1/z)] which is one-to-one outside C and which behaves like cz for large |z|, then $\tilde{\Omega}(w) = f^{-1}(w) + \left[1 \big/ f^{-1}(w)\right]$ is the flow round an obstacle whose boundary curve B is f(C).*" Actually, this procedure does work, but the application is a little tricky. Since f(z) = w = (1/2)(3z + 1/z), then $f^{-1}(w)$ = z = (1/3) (w + $\sqrt{w^2 - 3}$). Since $\Omega_d(z) = e^{-i\phi} z + \frac{e^{i\phi}}{z}$, we can write $\tilde{\Omega}(w) = \Omega_d(z) = e^{-i\phi} z + \frac{e^{i\phi}}{z}$ and substitute $f^{-1}(w)$ for z. Be sure to use the hint that (1/z) = (2w -3z). The result is

$$\tilde{\Omega}_{(+,-)}(w) = (1/3)\ (w + \sqrt{w^2 - 3}\ )\ e^{-i\phi} + \ (w - \sqrt{w^2 - 3}\ )\ e^{i\phi}$$

The subscripted signs on $\tilde{\Omega}_{(+,-)}(w)$ refer to the signs on the two square roots. If we allow for the fact that the square root may be positive or negative in sign, the first root in this equation might also be negative and the second could be positive. So we need a second equation:

$$\tilde{\Omega}_{(-,+)}(w) = (1/3)\ (w - \sqrt{w^2 - 3}\ )\ e^{-i\phi} + (w + \sqrt{w^2 - 3}\ )\ e^{i\phi}$$

Vasco noted (personal communication, Forum) that the solution would require using the correct value for the square roots in each quadrant. We find that we need only these two combinations of roots to plot the streamlines. Let $\Psi_{(\pm,\mp)}(u,v) = \mathrm{Im}[\tilde{\Omega}_{(\pm,\mp)}(w)]$. Then we find that $\Psi_{(+,-)}(u,v)$ must be plotted (using ContourPlot) over the right half plane, and $\Psi_{(-,+)}(u,v)$ must be plotted over the left half plane. Plotting without these constraints or with other combinations of roots produces plots that are not well suited to our purpose. One attempt using all four combinations of roots plotted the flow at the proper ϕ-angle outside the ellipse, but it added streamlines that appeared to be infinite and pointed in the direction of $3\pi/4$ and $-\pi/4$.

Out[2335]=

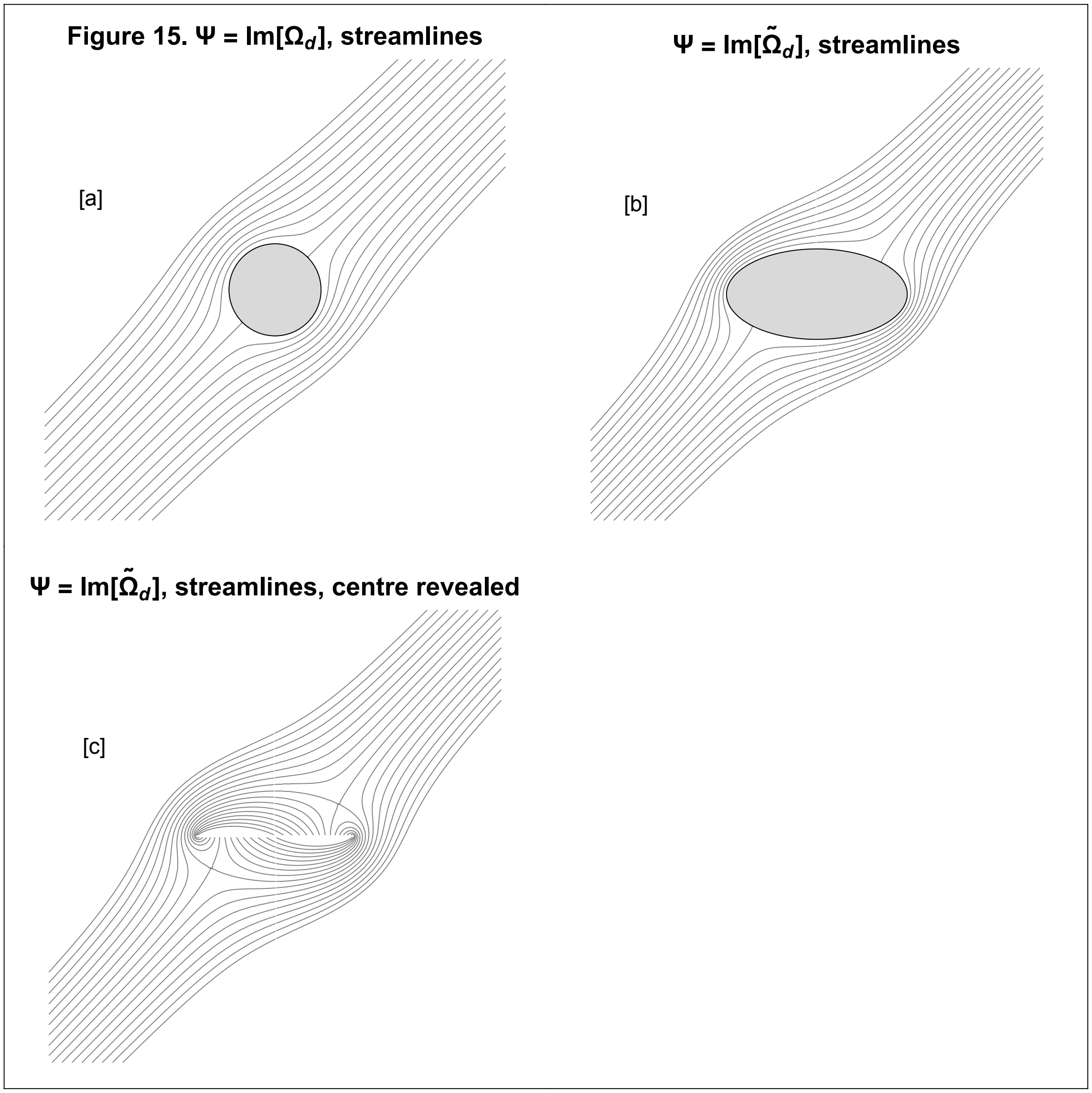

Figure 15. $\Psi = \mathrm{Im}[\Omega_d]$, streamlines