IV Geometry and Curvature of Harmonic Equipotentials

P. 520. Needham poses the problem for Section IV. *Given a family of curves $\mathcal{E}$ filling a region of the plane, how may we decide whether or not there exists a harmonic function Φ such that $\mathcal{E}$ is its family of equipotentials*. In other words, given a curve $\mathcal{E}_i$ that is a member of $\mathcal{E}$, is there a harmonic function Φ that has the same value on the whole of $\mathcal{E}_i$. What is the difference between Φ and $\mathcal{E}$? Since Φ can be assigned arbitrary values, why can we not just say that Φ is the family of equipotentials?

P. 520. Applying the test of p. 514 to the set of origin-centred circles with the rule Φ = |z| results in [4b]. This may be confusing because [4a] is not a set of origin-centred circles. The text is saying that both [4a] and a set of origin-centred circles with the rule Φ = |z| will produce something like [4b], so the test for harmonic Φ fails for both. But if the circles are assigned potentials with the rule Φ = ln|z|, the test succeeds on the circles, but probably not on [4a]. Evidently, the test applied to a similar set of concentric ellipses fails whether Φ = |z| or Φ = ln|z|.

P. 521, Regarding (9), is $\left(\frac{\partial_{\kappa_1}}{\partial_{s_1}} + \frac{\partial_{\kappa_2}}{\partial_{s_2}}\right)$ equivalent to $(\partial_{s_1}\kappa_1 + \partial_{s_2}\kappa_2)$? If so, why the change in notation?

Pp. 521-523.

- w = f(z) = $\Omega^{-1}(z)$
- z = Ω(w). The z-plane has grid of orthogonal straight line segments.
- w-plane is the image plane.
- $\mathcal{E}$ is in the w-plane (and so must $\mathcal{S}$ be).
- $\mathcal{E}$ is harmonic if and only if it is the image set of vertical lines in the z-plane.
- $\mathcal{E}$ has the potential function Φ(w) [not $\tilde{\Phi}$(w)?]
- The harmonic potential (Φ(w)) of $\mathcal{E}$ is the real part of Ω(w).

These definitions appear to invert the model presented on p. 501, which presents a mapping Ω: z ⟶ w and a real function Φ(z). In this model, the grid of vertical and horizontal lines is in the w-plane. This model is continued on p. 520. But it is still the case in both models that *the complex potential [of an irrotational and sourceless Pólya vector field] maps streamlines to horizontal lines and equipotentials to vertical lines.... Thus Ω is an analytical mapping*.

P. 522, ¶1. *The result (10) therefore implies that equation (9) is a necessary condition for $\mathcal{E}$ to be harmonic*. (9) implies that $\nabla \bullet \mathcal{K} = 0$, which means that $\mathcal{K}$ is sourceless. But why is sourceless $\mathcal{K}$ a necessary condition for $\mathcal{E}$ (rather than $\mathcal{S}$) to be harmonic? Does sourceless $\mathcal{K}$ imply that f is analytic? On p. 511, we learn that *If Φ is harmonic, then we know [(30), p. 500] that $\overline{H}$ will be sourceless, and so it will possess a stream function Ψ.*

Hence, we can say that sourceless $\mathcal{K}$ is a necessary condition for analytic $\overline{\mathcal{K}}$. Why do we want to know whether $\mathcal{E}$ is harmonic?

P. 523, top. The derivation of ∇|f'|. Begin with

$$\nabla|f'| = \frac{f'\overline{f''}}{|f'|}$$

$$|f'|\nabla|f'| = f'\,\overline{f''} \qquad \text{(multiply through by |f'|)}$$

This is what we now need to prove.

$$2|f'|\nabla|f'| = \nabla\,|\,f'\,|^2 = \nabla(f'\,\overline{f''}) \qquad (\text{LHS} \times 2)$$

$$= f'\nabla\overline{f'} + \overline{f'}\nabla f' = 2f'\,\overline{f''}$$

$$= f'(2\overline{f''}) + \overline{f'}\cdot 0 = 2f'\,\overline{f''}$$

(substitutions from p. 518, 522, bottom)

Hence $\nabla|f'| = \frac{f'\overline{f''}}{|f'|}$. The substitution of $2\overline{f''}$ for $\nabla\overline{f'}$ depends on $\overline{\nabla}f = 2f' \Longrightarrow \nabla\overline{f'} = 2\overline{f''}$, where f is analytic.

P. 523, "In order for $\mathcal{E}$ (or $\mathcal{S}$) to be harmonic, X must be divergence-free and curl-free." Referring back to subsection 2, p. 520, $\mathcal{E}$ and $\mathcal{S}$ are lines of equal potential Φ and equal flux density Ψ respectively. They are the curves on which the complex curvature is defined. Curves of $\mathcal{E}$ are orthogonal to curves of $\mathcal{S}$ just as the curves of Φ are orthogonal to those of Ψ if H is analytic. We can arbitrarily assign different values to Φ or Ψ. So it appears that they are the same thing. why do we have this new notation and what does the harmonicity of H have to do with the harmonicity of $\mathcal{K}$?

P. 523, ¶1. *In order for $\mathcal{E}$ (or $\mathcal{S}$) to be harmonic, X must be divergence-free and curl-free, so*

$$k_1 = \frac{\partial}{\partial_{s_2}} \ln\left|X\right| \quad \text{and} \quad k_2 = \frac{\partial}{\partial_{s_1}} \ln\left|X\right|,$$

from which (9) immediately follows. Where do these equations with ln|X| come from?

P. 524, ¶4. *(11) yields the desired formula (...) for the curvature of the streamlines and equipotentials of a sourceless, irrotational vector field in terms of its complex potential:*

$$k_1 + ik_2 = -i\ \frac{|\Omega'|\overline{\Omega''}}{(\overline{\Omega'})^2}$$

Where does this come from? It is just ($\Omega'\ \mathcal{K}_\Omega$) in the w plane. Begin with the equation on p. 523, line 4, and note that $\mathcal{K}_f$ is the default form of $\mathcal{K}$.

$$\mathcal{K} = \frac{i\,\overline{f''}}{\overline{f'}|f'|}$$

Then from (11)

$$\mathcal{K}_f = -\Omega'(\mathrm{w})\ \mathcal{K}_\Omega(w)$$

$$= -\Omega'(\mathrm{w})\ \frac{i\,\overline{\Omega''}}{\overline{\Omega'}|\Omega'|}$$

$$= -\mathrm{i}\frac{\overline{\Omega'}}{\overline{\Omega'}}\ \frac{\mathrm{i}\Omega'\,\overline{\Omega''}}{\overline{\Omega'}|\Omega'|}$$

$$= -\mathrm{i}\ \frac{|\Omega'|^2\overline{\Omega''}}{|\Omega'|(\overline{\Omega'})^2}$$

$$= -\mathrm{i}\ \frac{|\Omega'|\overline{\Omega''}}{(\overline{\Omega'})^2}$$

This is what we wanted.

p. 525, §§4 Further Geometry of the Complex Curvature

The streamlines of the complex curvature are the level curves of the amplification, |f'|.

Are these level curves the same that we former labeled Ψ? It appears that they are, because Figure [9a] shows ∇Ψ as a change from a in the position of a level curve |f'(z)| = |f'(a)|. These absolute value labels arranged on the curves are not themselves points in the curve. They are labels indicating curves whose points have the same absolute values. The vector arrow indicating $\mathcal{K}$ is tangent to the level curve, so the curves of [9a] must represent streamlines of the vector field $\mathcal{K}$ rather than streamlines of f. But the label Ψ is puzzling. This Ψ equals 1/|f'|, which is apparently the stream function for *the complex curvature* of any analytic mapping f, by (10), p. 521.

$$\Psi = \mathrm{F}[\mathcal{K}, \mathrm{C}] = \nabla \bullet \mathcal{K} = \nabla \bullet \frac{i\,\overline{f''}}{f'|f'|}$$

This Ψ is not the stream function for f. Previously, we have examined the streamlines of vector fields $\overline{H}$ or X on curves in the z or w-plane. In this case, we are examining the streamlines of the vector field $\mathcal{K}$ in the z-plane. In [9a] one might also expect to see a curve f, but we see only the level curves of |f'(z)|. This is perhaps confusing, because [9] shows the function f' from [9a] to [9b], which implies that [9a] is situated in the z-plane. We conclude that there is a convention of displaying the level values of a function (f or f') in the z-plane. [9b] might be referred to as "the f'(z) plane".

At this point, we wonder why the streamlines in [9a] are labeled "|f'(z)| = |f'(a)|", but the derivation on p. 526 requires Ψ = 1/|f'|. Shouldn't the streamlines also be labeled 1/|f'|, or is Ψ = 1/|f'| a different stream function from that in [9a]?

Line 5, p. 526 says "Thus, if Ψ is any decreasing function of |f'| then -i∇Ψ will be a source-less vector field in the direction of $\mathcal{K}$. How do we know this? Because $\mathcal{K}$ = -i∇Ψ points in the direction of $\overline{H}$ [(23), p. 497].

In ¶ 2, p. 526, $|\mathcal{K}|$ is derived from |∇Ψ|. This requires inspection of [9] and it uses items from both [9a] and [9b]. We are given equations for d|f'| and Ψ, but we need to obtain dΨ by inspection. Since dΨ = (negative change in |f'|; see [9a]), and d|f'| varies inversely with $|f'|^2$ (change in radius of f'(z)); see [9b]), we can write

$$\mathrm{d}\Psi = \frac{-d|f'|}{|f'|^2}$$

Vasco (personal communication, Forum) has pointed out that this equation can be derived more easily by basic differentiation. Since Ψ = 1/|f'|, $\mathrm{d}\Psi = -\left(1/|f'|^2\right)$ d|f'|.

The gradient requires dividing $\mathrm{d}\psi$ by $|\mathrm{dz}_2|$. Now, prefixing a minus sign to cancel the negative value of dΨ, we can write

$$|\nabla\Psi| = -\frac{\mathrm{d}\Psi}{|\mathrm{dz}_2|} = -\frac{-\frac{d|f'|}{|f'|^2}}{|\mathrm{dz}_2|} = \frac{d|f'|}{|f'|^2|\mathrm{dz}_2|}$$

$$= \frac{|f''(a)\,\mathrm{dz}_2|}{|f'|^2|\mathrm{dz}_2|} = \frac{|f''|}{|f'|^2} = |\mathcal{K}|$$

pp. 527-528.

S is used to denote the value of circulation round the boundary B in Figure [12] with an obstacle inserted into the flow. Then Needham states that the circulation round any simple loop enclosing the obstacle will also equal S, but the circulation around loops that do not enclose the obstacle will vanish. Why can he then say, “Put differently, this says that the flux crossing a path between two points in the flow is dependent on the choice of that path.” ?

The next sentence begins “The circulation $\psi(z)$ along a path from a fixed point to a variable point z is a multifunction of z:” The use of ψ for circulation is confusing. This simple has previously been used for flux. Did Needham perhaps intend to use $\phi(z)$? It is easy to confuse the two in typing.

[Note: This section has been rewritten in the new edition to correct this problem. See the correction in the Forum.]