Notes, Needham, Chapter 12
G. Palmer, December 5, 2018, 11 am PST

P. 517-518

On p. 517, Needham explains that “the gradient operator ∇ of vector calculus acts on a real function R(x,y) to produce the gradient vector field

$$\nabla \mathrm{R} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \mathrm{R} = \begin{pmatrix} \partial_x R \\ \partial_y R \end{pmatrix}, \qquad (1)$$

and we are free (as we have previously done) to think of this as a complex function

$$\nabla \mathrm{R} = (\partial_x R + \mathrm{i}\partial_y R).\text{”} \qquad (2)$$

Needham states beginning in line 12, p. 518 “However there would seem to be no natural way of applying ∇ directly to **f** to obtain a new vector field ∇**f** “, where **f** has previously been defined as the vector field

$$\mathbf{f} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Needham goes on to show that if **∇f** and f are replaced by their complex versions, then the “natural definition” consists in multiplying the two complex representations. This purpose of this note is to demonstrate that a matrix representation of the calculation can be deduced from the definition of the gradiant field of ψ presented on p. 500 and the equations (1) and (2) above.

$$\nabla\psi = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \psi = \begin{pmatrix} \partial_x \psi \\ \partial_y \psi \end{pmatrix} \Longleftrightarrow \nabla\psi = \partial_x \psi + \mathrm{i}\, \partial_y \psi. (3)$$

R in (1) and ψ in (3) are real functions. R(x,y) is specifically a function of x and y. I would argue that just as we are free to think of ∇R as a complex function, we should be free to think of ∇**f** as a complex function. On p. 210, we are given the following pertaining to the Jacobian and analytic functions.

$$\partial_x f = \partial_x u + \mathrm{i}\partial_x v$$

$$-\partial_y f = \partial_y v - \mathrm{i}\partial_y u$$

Hence, there seems no reason why we can not write

$$\nabla\mathbf{f} = \begin{pmatrix} \partial_x \boldsymbol{f} \\ \partial_y \boldsymbol{f} \end{pmatrix}$$

$$= \begin{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} \\ \\ \partial_y \begin{pmatrix} u \\ v \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \partial_x u \\ \partial_x v \end{pmatrix} \\ \begin{pmatrix} \partial_y u \\ \partial_y v \end{pmatrix} \end{pmatrix} \qquad (4)$$

$$\equiv \begin{pmatrix} \partial_x u \\ \partial_x v \end{pmatrix} + \mathrm{i} \begin{pmatrix} \partial_y u \\ \partial_y v \end{pmatrix}$$

$$\equiv (\partial_x u + \mathrm{i}\, \partial_x v) + \mathrm{i}\, (\partial_y u + \mathrm{i}\, \partial_y v)$$

$$= (\partial_x u - \partial_y v) + \mathrm{i}\, (\partial_x v + \partial_y u)$$

This is the result we wanted. It seems as much a “natural way” to calculate ∇f as the one presented by Needham. We take (4) to be a possible matrix representation of a complex number.

P. 518 [exercise]

The power of the complex gradient derives from the following fundamental result [exercise]:

A complex function f is analytic if and only if $\nabla f = 0$*, in which case we also have* $\overline{\nabla} f = 2 f'$*.*

On p. 210, we are given the following pertaining to the Jacobian and analytic functions. Needham notes that both expansion factors are equal.

$$f' = \partial_x u + \mathrm{i} \partial_x v = \partial_x f$$

$$f' = \partial_y v - \mathrm{i} \partial_y u = -\partial_y f$$

Given:

$$f = u + iv$$

$$\overline{\nabla} = \partial_x - \mathrm{i} \partial_y$$

Proof:

$$\overline{\nabla} f = (\partial_x - \mathrm{i} \partial_y)\,(u + iv)$$

$$= \partial_x u + \mathrm{i} \partial_x v - \mathrm{i} \partial_y u + \partial_y v$$

$$= (\partial_x u + \mathrm{i} \partial_x v) + (\partial_y v - \mathrm{i} \partial_y u)$$

$= \partial_x f - \partial_y f = 2f'$

This is what we wanted.

P. 519 [exercise]

Since w = f(z), a straightforward application of the chain rule yields [exercise]

$$\nabla_z = \overline{f'}\nabla_w \quad \textit{and} \quad \overline{\nabla}_z = f'\overline{\nabla}_w \qquad (8)$$

I will refer to these as (8a) and (8b). I could not make sense of this. Vasco has provided a derivation of the first equation. This is my attempt to duplicate it with commentary.

Let z = x + iy, w = u + iv = f(z), $\nabla_z = \partial_x + i\,\partial_y$, and $\nabla_w = \partial_u + i\,\partial_v$.

Vasco introduced a new element into the interpretation of ∇_z and ∇_w, i.e. that they apply to x, which is a real function like ϕ or $\tilde{\Phi}$. He wrote "Let x(z) = $\tilde{x}$(w) be such a function." We can now rewrite (8) as

$$\nabla_z x(z) = \overline{f'}\nabla_w \tilde{x}(w) \quad \text{and} \quad \overline{\nabla}_z x(z) = f'\overline{\nabla}_w \tilde{x}(w) \qquad (1)$$

The task of deriving (1a) is to begin with the $\nabla_z x(z)$ and derive $\overline{f'}\nabla_w \tilde{x}(w)$. In doing so, we rewrite the function of x and y on the LHS as a function of u and v on the RHS. Then, the same will be done for (1b). One may ask, since we already have the function f from the z-plane to the w-plane, why interpose two new functions x and $\tilde{x}$? The answer seems to be that f is a complex function, but we require real functions that stand in for the potential function of the complex potential.

Vasco begins "Since w = f(z) = u + iv we can use the chain rule and write $\partial_x x = \partial_x u \partial_u \tilde{x} + \partial_x v \partial_v \tilde{x}$ [and similarly for $\partial_y x$]." I'm suggesting a change of style: By reversing the order of the partials we can make the role of the chain rule more obvious. My reason for the reversal is an analogy. When the chain rule is applied, the partial should have a parallel structure to the usual arrangement of derivatives. If we are taking the derivative of $\tilde{x}$(w(z)), for example, the outer derivative $\frac{d\tilde{x}}{\mathrm{dw}}$ normally appears before the inner $\frac{\mathrm{dw}}{\mathrm{dz}}$, so one would see terms such as $\frac{d\overline{x}}{\mathrm{du}} \cdot \frac{\mathrm{du}}{\mathrm{dx}}$, which after cancelling $\frac{\mathrm{du}}{\mathrm{du}}$ becomes $\frac{d\overline{x}}{\mathrm{du}} \approx \partial_x x$. Corresponding to $\frac{d\overline{x}}{\mathrm{du}} \cdot \frac{\mathrm{du}}{\mathrm{dx}}$, we have $\partial_u \tilde{x} \partial_x u$, which is arranged in the same order, but does not cancel out in the same way.

Parallel structures for $\partial_x \tilde{x}$ and $\partial_y \tilde{x}$ would be

$$\partial_x x = \partial_u \tilde{x} \partial_x u + \partial_v \tilde{x} \partial_x v$$

$$\partial_y x = \partial_u \tilde{x} \partial_y u + \partial_v \tilde{x} \partial_y v$$

Derivation of $\nabla_z = \overline{f'}\nabla_w$

Then

$$\nabla_z x = \partial_x x + \mathrm{i}\, \partial_y x$$

$$= \partial_u \tilde{x} \partial_x u + \partial_v \tilde{x} \partial_x v + \mathrm{i}(\partial_u \tilde{x} \partial_y u + \partial_v \tilde{x} \partial_y v)$$

Now we note that $\nabla_w = \partial_u + i\, \partial_v$, which we want on the RHS, so we rearrange and separate the equation to obtain the components ∇_w and $\overline{f'}$. For the derivation of $\overline{f'}$, see (6), (7), (8) on p. 210, which contain the Cauchy-Riemann equations and two forms of f’ with f defined as u + iv, as it is here.

$$= (\partial_u \tilde{x} \partial_x u + \mathrm{i} \partial_u \tilde{x} \partial_y u) + (\partial_v \tilde{x} \partial_x v + \mathrm{i} \partial_v \tilde{x} \partial_y v)$$

$$= (\partial_x u + \mathrm{i} \partial_y u)\, \partial_u \tilde{x} + (\partial_x v + \mathrm{i} \partial_y v)\, \partial_v \tilde{x}$$

$$= (\partial_x u + \mathrm{i} \partial_y u)\, \partial_u \tilde{x} + (-\,\mathrm{i}\, \partial_x v + \partial_y v)\, i\, \partial_v \tilde{x}$$

$$= (\partial_x u - \mathrm{i} \partial_x v)\, \partial_u \tilde{x} + (\mathrm{i}\, \partial_y u + \partial_y v)\, i\, \partial_v \tilde{x}$$

$$= \overline{f'} \partial_u \tilde{x} + \overline{f'}\, i\, \partial_v \tilde{x}$$

$$= \overline{f'}\, (\partial_u \tilde{x} + i\, \partial_v \tilde{x})$$

$$= \overline{f'} \nabla_w \tilde{x}$$

Hence, we have shown that $\nabla_z = \overline{f'} \nabla_w$. It appears that this result hinges on the equality x(z) = $\tilde{x}$(w).

Derivation of $\overline{\nabla}_z = f' \;\; \overline{\nabla}_w$ (8b)

We introduce a change in notation. Since f(z) = w = u + iv, we dispense with f and write w(z) = u(z) + iv(z). Then we can write

$$\overline{\nabla}_z = w' \overline{\nabla}_w$$

but to highlight the role of the chain rule, we write

$$\overline{\nabla}_z = \overline{\nabla}_w w'$$

indicating that $\overline{\nabla}_w$ serves as the outer derivative and w' serves as the inner derivative. Now we expand with ϕ and $\tilde{\Phi}$, because we are more likely to encounter these than x and $\tilde{x}$. Since all the partials are real numbers, there is no need to apply the conjugate sign to the partial itself when it is part of a conjugate complex number.

$$\partial_x \Phi(z) = \partial_u \tilde{\Phi}\, \partial_x u + \partial_v \tilde{\Phi}\, \partial_x v$$

$$\partial_y \Phi(\mathrm{z}) = \partial_u \tilde{\Phi}\, \partial_y u + \partial_v \tilde{\phi}\, \partial_y v$$

We wish to show that

$$\overline{\nabla}_z \Phi(\mathrm{z}) = \overline{\nabla}_w(\tilde{\Phi}(\mathrm{w}))\, w'$$

$$\overline{\nabla}_z \Phi(\mathrm{z}) = \partial_x \phi(z) - i\partial_y \phi(\mathrm{z})$$

$$= \left(\partial_u \tilde{\Phi}\, \partial_x u + \partial_v \tilde{\Phi}\, \partial_x v\right) - \mathrm{i}\, (\partial_u \tilde{\Phi}\, \partial_y u + \partial_v \tilde{\phi}\, \partial_y v)$$

$$= \left(\partial_u \tilde{\Phi}\, \partial_x u - \mathrm{i}\, \partial_u \tilde{\Phi}\, \partial_y u\right) + (\partial_v \tilde{\Phi}\, \partial_x v - \mathrm{i}\, \partial_v \tilde{\phi}\, \partial_y v)$$

$$= \partial_u \tilde{\Phi}\, (\, \partial_x u + \mathrm{i}\, \partial_x v) + \partial_v \tilde{\Phi}\, (\, \partial_x v - \mathrm{i}\, \partial_y v)$$

$$= \partial_u \tilde{\Phi}\, (\, \partial_x u + \mathrm{i}\, \partial_x v) - \mathrm{i}\, \partial_v \tilde{\Phi}\, (\, i\, \partial_x v + \partial_x u) \text{ [C-R Eq.]}$$

$$= \left(\partial_u \tilde{\Phi} - \mathrm{i}\, \partial_v \tilde{\Phi}\right) \mathrm{w}'$$

$$= \overline{\nabla}_w(\tilde{\Phi})\, w'$$

or, since w’ is outside the derivative operator,

$$\overline{\nabla}_z = w'\, \overline{\nabla}_w \qquad \text{(8b) rewritten}$$

This is what we wanted. We can similarly rewrite (8a):

$$\nabla_z = \overline{w}'\, \nabla_w \qquad \text{(8a) rewritten}$$

P. 519, $\nabla_z \overline{\nabla}_z \Phi(\mathbf{z}) = |\, w'\, |^2\, \nabla_w\, \overline{\nabla}_w\, \tilde{\Phi}(\mathbf{w})$

Note that Vasco has pointed out that the LHS does not contain f(z), which appears in the text.

We return now to the problem of showing that

$$\nabla_z \overline{\nabla}_z \Phi(\mathrm{z}) = |\, f'\, |^2\, \nabla_w\, \overline{\nabla}_w\, \tilde{\Phi}(\mathrm{w}) \qquad \text{[tilde added on RHS]}$$

Which we rewrite leaving the arguments of Φ and $\tilde{\Phi}$ as understood with z on the LHS and w on the RHS.

$$\nabla_z \overline{\nabla}_z \Phi = |w'|^2 \nabla_w \overline{\nabla}_w \tilde{\Phi}$$

Following the steps at the bottom of p. 519,

$$\nabla_z \overline{\nabla}_z \Phi = \nabla_z \{f' \, \overline{\nabla}_w \tilde{\Phi}\} \quad \text{[corrected from } \Phi[f(z)] \text{ and } \Phi(w)]$$

We rewrite substituting w' for f'.

$$\nabla_z \overline{\nabla}_z \Phi = \nabla_z \{w' \, \overline{\nabla}_w \tilde{\Phi}\}$$

$$= \nabla_z w' \, \overline{\nabla}_w \tilde{\Phi} \; + \; w' \, \nabla_z \overline{\nabla}_w \tilde{\Phi} \quad \text{[differentiation of product rule]}$$

By the statement on p. 518, we know that $\nabla_z w' = 0$, because ∇_z is equivalent to the $\nabla = (\partial_x + i\,\partial_y)$ discussed there and our w is equivalent to the f in the statement. Since w is analytic, so is w'. Then

$$\nabla_z \overline{\nabla}_z \Phi = w' \, \nabla_z \overline{\nabla}_w \tilde{\Phi}$$

$$= w' (\overline{w}' \, \nabla_w) \, \overline{\nabla}_w \tilde{\Phi} \quad \text{[substituting from (8a), rewritten]}$$

$$= \; |w'|^2 \, \nabla_w \overline{\nabla}_w \tilde{\Phi}(w)$$

This is what we wanted.