P. 516, SS2, ln-3 to end. “Having made the connection with the toy model, we leave the remaining details to the interested reader.”

Given that

$$V(z) = [\nabla \bullet V\,(z_0)]\,\frac{(z-z_0)}{2} + Y(z),$$

where Y is sourceless and irrotational, the local behavior of the potential is

$$\Phi(z) = \frac{1}{4}[\nabla \bullet V\,(z_0)]r^2 + \Upsilon(z) \qquad (6)$$

To obtain (6),

$$\nabla\Phi = \nabla \bullet V\,(z_0)\,\frac{(z-z_0)}{2} + Y(z)$$

Needham makes an explicit connection with the toy model and we want to obtain an equation for $\Phi(z)$. On p. 515, we have

$$\Phi(z) = \frac{1}{4}\ S\ |z|^2$$

Substitute $\nabla \bullet V\,(z_0)$ for S and r for |z|. Then adding $\Upsilon(z)$ for 0, the result is (6).

P. 516-517, SS 3, [Exercise]. The Meaning of the Laplacian

In fact if you have already convinced yourself of (6) then [exercise] you may derive (7) by using the fact that the harmonic function Υ obeys Gauss’ Mean Value Theorem.

We must derive

$$\langle\Phi\rangle - \Phi(p) = \frac{1}{4}\, r^2 \Delta\Phi$$

We know that $\Delta\Phi = \nabla \bullet V = \nabla \bullet \nabla\Phi$ (pp. 511, 516, 517).

From (6)

$$\Phi(p) = \frac{1}{4}[\nabla \bullet V(p)]r^2 + \Upsilon(p)$$

$$\langle\Phi\rangle = \langle\frac{1}{4}[\nabla \bullet V]r^2 + \Upsilon\rangle$$

$$\langle\Phi\rangle - \Phi(p) = \langle\frac{1}{4}\,\Delta\Phi r^2\rangle - \frac{1}{4}\,\Delta\Phi(p)r^2 + (\langle\Upsilon\rangle - \Upsilon(p))$$

$$= \frac{1}{4}r^2[<\Delta\Phi> - \Delta\Phi(p)] + 0$$

$$<\Phi> - \Phi(\mathrm{p}) = \frac{1}{4}r^2\Delta\Phi$$

This is what we wanted, but I don't understand how we reduce [<ΔΦ> - ΔΦ(p)] to ΔΦ in the last step. But see the next exercise.

P. 517, [Exercise] *Let V = ∇Φ be the vector field of the potential function Φ. The flux of V out of a (non-infinitesimal) circle C of radius r is then [Exercise].*

$$F[V, C] = 2\pi r\, \partial_r(\Phi)$$

We use a model similar to that of [18], p. 496. The lines of flux are rays and the circle forms a line of constant potential. We can draw one radius on the real line. Then N = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and r = x. The total flux of V out of C is the same no matter the size of the circle, so the total is the sum of the fluxes at all points on the circle. Since Φ is harmonic, the average value is Φ(p).

$$\mathrm{F[V, C]} = \int_C \mathrm{V} \bullet \mathrm{N}\, ds$$

$$= 2\pi \mathrm{r}\ \mathrm{V}\bullet\mathrm{N} = 2\pi\mathrm{r}\ (\nabla\Phi \bullet \mathrm{N})$$

$$= 2\pi\mathrm{r} \begin{pmatrix} \partial_x\Phi \\ \partial_y\Phi \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= 2\pi\mathrm{r}\ \partial_x\Phi(p) = 2\pi\mathrm{r}\ \partial_r <\Phi> \qquad (1)$$

This is what we wanted.

Now suppose that V is not sourceless, but that its flux density ∇•V = ΔΦ is constant. Gauss' Divergence Theorem [(7), p. 481] then yields $F[V, C] = \pi r^2\, \Delta\Phi$ *. Inserting this into the previous result, we find that*

$$\partial_r < \Phi > = \frac{1}{2} r\, \Delta\Phi$$

$$\int\!\!\int_C [\nabla\bullet\mathrm{V}]\, dA \qquad \text{[apply Gauss' divergence theorem]}$$

$$= \Delta\Phi \int\!\!\int_C dA \qquad \text{[because } \nabla\bullet\mathrm{V} = \Delta\Phi \text{ is constant]}$$

$$= \pi\mathrm{r}^2\, \Delta\Phi \qquad (2)$$

$$F[V, C] = 2\pi r\, \partial_r < \Phi > = \pi r^2\, \Delta\Phi \qquad [(1) = (2)]$$

$$\Longrightarrow 2\partial_r < \Phi > = r\Delta\Phi$$

$$\partial_r < \Phi > = \frac{1}{2}\, r\, \Delta\Phi$$

This is what we wanted.

Which may be integrated to yield the formula in (7).

$$\int \partial_r < \Phi > dr = \int \frac{1}{2}\, r\, \Delta\Phi\, dr + \text{const.}$$

$$<\Phi> = \frac{1}{4}\, r^2 \Delta\Phi + \Phi(p)$$

$$<\Phi> - \Phi(p) = \frac{1}{4}\, r^2 \Delta\Phi$$

This is what we wanted.