Notes on Needham, Visual Complex Analysis, Chapter 12, pp. 515...
G. Palmer, December 23, 2018, 3:15 pm PST

P. 513. *For example, if $\Phi = \cos[\cos x \sinh y]\, e^{\sin x \cosh y}$ then [exercise]*

$$\overline{\nabla \Phi}(x,y) = \cos[\cos x \sinh y]\, e^{\sin x \cosh y} \cos x \cos h\, y + F,$$

where F stands for three terms which vanish when y = 0. Using (1) we get $\Omega'(z) = e^{\sin z} \cos z$, and hence $\Psi = Im\, e^{\sin z}$.

Find $\overline{\nabla\Phi}$(x, y).

$$\nabla\Phi(x, y) = \nabla\,(\cos[\cos x \sinh y]\, e^{\sin x \cosh y})$$

$$= e^{\sin x \cosh y}\, \partial_x \cos[\cos x \sinh y] \;+\; \cos[\cos x \sinh y]\, \partial_x\, e^{\sin x \cosh y}$$

$$+\, i\,(e^{\sin x \cosh y}(\partial_y \cos[\cos x \sinh y] \;+\; \cos[\cos x \sinh y]\, \partial_y\, e^{\sin x \cosh y})$$

$$= -e^{\sin x \cosh y} \sin[\cos x \sinh y]\, \partial_x [\cos x \sinh y] \;+\; \cos[\cos x \sinh y] \cos x\, \cosh y\; e^{\sin x\, \cosh y}$$

$$+\, i\,(-e^{\sin x \cosh y} \sin[\cos x \sinh y]\, \partial_y [\cos x \sinh y] \;+\; \cos[\cos x \sinh y] \sin x\, \sinh y\; e^{\sin x\, \cosh y})$$

$$=\; e^{\sin x \cosh y} \sin[\cos x \sinh y] \sin x\, \sinh y \;\;+\; \cos[\cos x \sinh y] \cos x\, \cosh y\; e^{\sin x\, \cosh y}$$

$$+\, i\,(-e^{\sin x \cosh y} \sin[\cos x \sinh y] \cos x\, \cos h\, y \;+\; \cos[\cos x \sinh y] \sin\, x\, \sinh y\; e^{\sin x\, \cosh y})$$

Then

$$\overline{\nabla\Phi}(x, y) = e^{\sin x \cosh y} \sin[\cos x \sinh y] \sin x\, \sinh y \;\;+\; \cos[\cos x \sinh y] \cos x\, \cosh y\; e^{\sin x\, \cosh y}$$

$$+\, i\,(e^{\sin x \cosh y} \sin[\cos x \sinh y] \cos x\, \cos h\, y \;-\; \cos[\cos x \sinh y] \sin\, x\, \sinh y\; e^{\sin x\, \cosh y})$$

$$= \cos[\cos x \sinh y]\, e^{\sin x\, \cosh y} \cos x\, \cosh y \;+ F \qquad \text{(Where F = 1st, 3rd and 4th terms)}$$

The terms constituting F go to zero, so this is what we wanted.

When y = 0, sin y = 0, sinh y = 0, cos y = 1, cosh y = 1.

$$\overline{\nabla\Phi}(x, y) = \;\; \cos[0] \cos x \cdot 1 \cdot e^{(\sin x) \cdot 1}$$

$$= \cos x\;\, e^{\sin x}$$

Substitute z for x.

$$\Omega'(z) = e^{\sin z} \cos z$$

$$\Psi = \text{Im} \int_z \Omega'(z)\, dz \; = \text{Im}\, e^{\sin z}$$

P. 515, 2. Conformal Invariance of the Laplacian

We are trying to prove (4):

$$\Delta\Phi(\text{z}) = \left[\Delta\, \tilde{\Phi}(w)\right] |\, f'(z)\,|^2 \qquad (4)$$

We note that $\Delta\Phi$ is the Laplacian $\nabla\bullet\nabla\Phi$, which is here apparently $\nabla\bullet$V. ($\nabla\Phi$ is normally defined as $\overline{H}$, the *Pólya* field, as in (28), p. 500. Judging from the discussion in *1. Conformal Invariance and Harmony*, pp. 513-514, Φ is the real part of an *analytic* function Ω.) Needham wants to "rephrase the result in terms of flux densities." The "rephrase" or rewrite is

$$\nabla\bullet\text{V}(\text{z}) = [\nabla\bullet\tilde{V}(\text{w})]\;\; |\, f'(z)\,|^2 \qquad (5)$$

In ¶ 3, the givens and algebra lead to

$$\tilde{\Phi}(\text{w}) = \Phi(\text{z}) = \tfrac{1}{4}\text{S}\;|\,\text{z}\,|^2 \;=\; \tfrac{1}{4}\tfrac{S}{|c|^2}\;|\,\text{w}\,|^2 \qquad ((1))$$

Needham concluded that "The field $\tilde{V} = \nabla\tilde{\phi}$ therefore has uniform flux density $\nabla \bullet \tilde{V} = S/|\,c\,|^2$". Where does this equation come from? Calculate $\nabla\bullet$V to check Needham's value. Then calculate $\nabla\bullet\tilde{V}(\text{w})$.

$$\Phi(\text{z}) = (S/4)\,|\,z\,|^2$$

$$\text{V} = \nabla\Phi(\text{z}) = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} [(\text{S}/4)\left(x^2 + y^2\right)]$$

$$= (\text{S}/4)\begin{pmatrix} 2\,x \\ 2\,y \end{pmatrix} = (\text{S}/2)\text{z}$$

$$\nabla\bullet\text{V} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \bullet (\text{S}/4)\begin{pmatrix} 2\,x \\ 2\,y \end{pmatrix}$$

$$= (\text{S}/4)\;(2 + 2) = \text{S}$$

Vasco simplifies the calculation of the Laplacian using $\partial_u^2\;|\,\text{z}\,|^2 = 2 = \partial_u^2\;|\,\text{z}\,|^2$. Then

$$\nabla\bullet\text{V} = (\text{S}/4)\;(\partial_x^2\;|\,\text{z}\,|^2 + \;\partial_y^2\;|\,\text{z}\,|^2) = \;(\text{S}/4)\;4 = \text{S}$$

We calculate $\nabla\bullet\tilde{V}(\text{w})$. Let w = u + iv.

$\tilde{\Phi}(\mathrm{w}) = \frac{1}{4}\frac{S}{|c|^2} \;|\; \mathrm{w} \;|^2$

$\nabla \bullet \tilde{V} = \frac{1}{4}\frac{S}{|c|^2}\, 4 \quad = \frac{S}{|c|^2}$

(Δ is the same for $|w|^2$ as for $|z|^2$)

For our own interest, we calculate $\tilde{V}$.

$\tilde{V} = \nabla\tilde{\Phi}(\mathrm{w}) = \begin{pmatrix} \partial_u \\ \partial_v \end{pmatrix} [\frac{S}{4|c|^2}\left(u^2 + v^2\right)]$

$= \frac{S}{4|c|^2}\begin{pmatrix} 2\,u \\ 2\,v \end{pmatrix} = \frac{S}{2|c|^2}\mathrm{w} = \frac{\mathrm{Sc}}{2|c|^2}\mathrm{z}$

Return to (5), calculate $|f'(z)|^2$, and substitute the calculated flux densities.

$|f'(z)| = |(cz)'| = |c|$

$\mathrm{S} = \left(S/|c|^2\right)\;|f'(z)|^2$

$= (S/|c|^2)\;|c|^2 \;=\; S$

This proves (5), which is a rewrite of (4), so it also proves (4) for this case.