Needham, Chapter 12, Exercise 9, p. 570.
G. Palmer, February 11, 2019, 8:15 am PST

9 From Ex. 7(i) we see that $\Delta = \overline{\nabla}\nabla = 4\partial_z\partial_{\bar{z}}$. Use this fact to solve the following problems:

(i) Show that $\Phi = \left[1-\left(x^2+y^2\right)\right]^{-1}$ satisfies $\Delta\Phi = 4\,\Phi^2(2\Phi - 1)$.

Rewrite $\Phi = [1 - z\,\bar{z}]^{-1}$. Note that $\partial_z\Phi = \bar{z}\,\Phi^2$, $\partial_{\bar{z}}\Phi = z\,\Phi^2$, and $z\,\bar{z} = \frac{\Phi-1}{\Phi}$.

$\Delta\Phi = 4\partial_z\partial_{\bar{z}}\Phi = 4\,\partial_z[\mathrm{z}\,\Phi^2]$ [subst. for $\partial_{\bar{z}}\Phi$]

$= 4\,(\Phi^2 + 2\mathrm{z}\,\Phi\,\partial_z\Phi)$ [product rule derivatives]

$= 4\,(\Phi^2 + 2\,\mathrm{z}\bar{z}\,\Phi^3)$ [subst. for $\partial_z\Phi$]

$= 4\,\Phi^2\,(1 + 2\,\frac{\Phi-1}{\Phi}\,\Phi) = 4\,\Phi^2\,(2\phi - 1)$ [subst. for $\mathrm{z}\bar{z}$ & rearrange]

We have shown that $\Phi = \left[1-\left(x^2+y^2\right)\right]^{-1}$ satisfies $\Delta\Phi = 4\,\Phi^2(2\Phi - 1)$.

(ii) Solve $\Delta F = e^{\bar{z}}$ for F by formally integrating with respect to z and then with respect to $\bar{z}$. Deduce that $R = \frac{1}{4}\,e^x\,(x\cos y + y\sin y)$ is a solution of $\Delta R = e^x\cos y$. Verify this by calculating ΔR explicitly.

$\Delta\mathrm{F} = 4\partial_z\partial_{\bar{z}}\mathrm{F} = e^{\bar{z}}$

$\mathrm{F} = \frac{1}{4}\int\int e^{\bar{z}}\,dz\,d\bar{z} = \frac{1}{4}\left[\int e^{\bar{z}}\,d\bar{z} + f(\bar{z})\ \right] = \frac{1}{4}\,([\bar{z}\,e^{\bar{z}} + \int f(\bar{z})]$

Deduce that $R = \frac{1}{4}\,e^x\,(x\cos y + y\sin y)$ is a solution of $\Delta R = e^x\cos y$.

$\Delta\mathrm{F} = e^{\bar{z}} = e^x\,e^{\mathrm{iy}} = e^x\,(\cos \mathrm{y} + \mathrm{i}\sin \mathrm{y})$

Let $f(\bar{z}) = 0$. Then

$\mathrm{F} = \frac{1}{4}\,([\bar{z}\,e^{\bar{z}} + \int f(\bar{z})] = \frac{1}{4}\,\bar{z}\,e^x\,(\cos \mathrm{y} + \mathrm{i}\sin \mathrm{y})$

$= \frac{1}{4}\,(\mathrm{x} - \mathrm{iy})\,e^x\,(\cos \mathrm{y} + \mathrm{i}\sin \mathrm{y}) = \frac{1}{4}\,[\,\mathrm{x}\,e^x\,(\cos \mathrm{y} - \mathrm{iy}\cos \mathrm{y} + \mathrm{i}\,\mathrm{x}\sin \mathrm{y} + \mathrm{y}\sin \mathrm{y})]$

$= \frac{1}{4}\,e^x\,[(\mathrm{x}\cos \mathrm{y} + \mathrm{y}\sin \mathrm{y}) + \mathrm{i}\,(\,\mathrm{x}\sin \mathrm{y} - \mathrm{y}\cos \mathrm{y})]$

Then

$$\mathrm{Re}(F) = \frac{1}{4}\, e^x\,(x \cos y + y \sin y) = \frac{1}{4} \int\!\!\int e^x \cos y \, dz \, d\bar{z} = \frac{1}{4} \int\!\!\int \mathrm{Re}[e^z] \, dz \, d\bar{z}$$

This implies that $R = \frac{1}{4}\, e^x\,(x \cos y + y \sin y)$ is a solution of $\Delta R = e^x \cos y$. The proof depends on the linearity of the integral on a complex number, i.e. $\int f = \int u(x, y) + i \int v(x, y)$.

Verify this by calculating ΔR explicitly.

If R is a solution of ΔR, then show that

$$4 \partial_z \partial_{\bar{z}} R = 4\, \partial_z \partial_{\bar{z}}[\tfrac{1}{4}\, e^x\,(x \cos y + y \sin y)] = e^x \cos y$$

From exercise 7, we have

$$\partial_z = \frac{1}{2}\,(\partial_x - i\,\partial_y) \qquad \text{and} \qquad \partial_{\bar{z}} = \frac{1}{2}\,(\partial_x + i\,\partial_y)$$

Then, canceling 4 in numerator and denominator, we have

$$\partial_{\bar{z}}[e^x(x \cos y + y \sin y)] = \frac{1}{2} \cdot \partial_x[e^x\,(x \cos y + y \sin y)] + i\ \partial_y[\, e^x(x \cos y + y \sin y)]$$

$$= \frac{1}{2}\,[e^x\,((\cos y + x \cos y + y \sin y) + i\,(\sin y - x \sin y + y \cos y))]$$

$$\partial_z\left[\frac{1}{2}\, e^x \left((\cos y + x \cos y + y \sin y) + i\,(\sin y - x \sin y + y \cos y)\right)\right]$$

$$= \frac{1}{2} \cdot \frac{1}{2}\, \partial_x[\, e^x\,((\cos y + x \cos y + y \sin y) + i\,(\sin y - x \sin y + y \cos y))]$$

$$+ i\, \partial_y[\, e^x\,((\cos y + x \cos y + y \sin y) + i\,(\sin y - x \sin y + y \cos y))]$$

$$= \frac{1}{4} \cdot 4\, e^x \cos y = e^x \cos y$$

This is what we wanted. This computation was computer-assisted. Note that Vasco has written a shorter version by substituting R for $\frac{1}{4} e^x(\, x \cos y + y \sin y)$.

(iii) Show that if f is the most general harmonic function in the plane, then $f(z, \bar{z}) = p(z) + q(\bar{z})$, where p and q are arbitrary analytic functions.

After several rereadings and discussions with Vasco, I read this exercise to mean that the most general harmonic function f can be written as $f(z, \bar{z}) = p(z) + q(\bar{z})$. Vasco put it this way: “we have to **find** the most general harmonic function and **show** that it can be written in the form given” (personal communication, Forum). Why does the general harmonic function work with z and $\bar{z}$?

Perhaps because the Laplacian operator can be written $\partial_x^2 + \partial_y^2 = \Delta = \nabla\overline{\nabla} = \overline{\nabla}\nabla = 4\partial_z\partial_{\bar{z}}$, which shows that it can be written either in x and y or in z and $\bar{z}$. The definition of harmonic is based on the Laplacian, so we must work with one of these pairs of variables. Perhaps an easier way to think about this is to treat it as a criterion for a function to be harmonic. I would state the criterion as follows: *A function in z and $\bar{z}$ is harmonic only if its part or parts each contain a single variable, either z or $\bar{z}$.* Then it follows that all such parts can be grouped into the two parts that are here named p(z) and q($\bar{z}$).

The proof

Let F be any harmonic function in z and $\bar{z}$. Then, F is the solution to the following:

$$4\partial_z\partial_{\bar{z}}F = 0$$

which implies that

$$\partial_z\partial_{\bar{z}}F = 0$$

Note that the order of the variables in the Laplacian --- $\partial_z\partial_{\bar{z}}$ versus $\partial_{\bar{z}}\partial_z$ --- is reversible (commutative), so we use them in the order that is convenient to the proof.

The equation is true only if $\partial_{\bar{z}}F$ is a function with a single variable $\bar{z}$. It can't have both. Call the function Q($\bar{z}$). The same constraint is true if we write $\partial_{\bar{z}}\partial_z F = 0$. Now the function $\partial_z F$ must have the single variable z. Call the function P(z).

Finding q($\bar{z}$)

In order to integrate Q($\bar{z}$), stipulate that it is analytic and find $q(\bar{z}) = \int Q(\bar{z})d\bar{z}$, which is a solution (a harmonic F) and also a function of $\bar{z}$ only.

Finding p(z)

Stipulate that $P(z) = \partial_z F$ is analytic and find $p(z) = \int P(z)dz$, which is another solution.

Since p(z) and q($\bar{z}$) are both solutions of ΔF, p(z) + q($\bar{z}$) is also a solution.

We have shown that *if f is the most general harmonic function in the plane, then $f(z, \bar{z}) = p(z) + q(\bar{z})$, where p and q are arbitrary analytic functions.*

An alternative proof

Let F be any harmonic function in z and $\bar{z}$. Then, F is the solution to the following:

$$4\partial_z\partial_{\bar{z}}F = 0$$

which implies that

$$\partial_z \partial_{\bar{z}} F = 0$$

Note that the order of the variables in the Laplacian --- $\partial_z \partial_{\bar{z}}$ versus $\partial_{\bar{z}} \partial_z$ --- is reversible (commutative), so we use them in the order that is convenient to the proof.

The equation is true only if $\partial_{\bar{z}} F$ is a function with a single variable $\bar{z}$. It can't have both. Call the function q($\bar{z}$). The same constraint is true if we write $\partial_{\bar{z}} \partial_z F = 0$. Now the function $\partial_z F$ must have the single variable z. Call the function p(z). Notice also that q($\bar{z}$) and p(z) must be analytic in order to assure that we can differentiate them. Then ∂_zp(z) and $\partial_{\bar{z}}$q($\bar{z}$) are also single-valued functions in z or $\bar{z}$. Hence, p(z) and q($\bar{z}$) are solutions to ΔF = 0.

Since p(z) and q($\bar{z}$) are both solutions of ΔF, p(z) + q($\bar{z}$) is also a solution.

We have shown that *if f is the most general harmonic function in the plane, then f(z, $\bar{z}$) = p(z) + q($\bar{z}$), where p and q are arbitrary analytic functions*.

A General Proof

We can generalize the proof as follows: If we write $\partial_Z F$, where Z = (z or $\bar{z}$), and we stipulate that $\partial_Z F$ is analytic with a single variable, then a function in the <u>same</u> variable (z or $\bar{z}$) as that in $\partial_Z F$ can be found by either differentiating or integrating $\partial_Z F$ on that variable, or by using it "as is". This new function $\left(..., \int \partial_Z F \, dZ,\ \partial_Z F,\ \partial_Z^2 F,\ ...\right)$ is a solution to the harmonic equation, because the harmonic equation requires that this new function be differentiated on the complement variable to obtain 0. If two possible new solutions (i.e. created under these constraints), one in z and the other in $\bar{z}$, are written p(z) and q($\bar{z}$), then we can write *f(z, $\bar{z}$) = p(z) + q($\bar{z}$)*.

Review of exercise 9

In part (i) we got some practice with the algebra of $4\partial_z \partial_{\bar{z}}$. In part (ii) we learned to find a solution to an equation containing $4\partial_z \partial_{\bar{z}}$. In part (iii) we found a solution for the most general harmonic function in z and $\bar{z}$.