Needham, Chapter 12, Exercise 8, p. 570.
G. Palmer, January 26, 2019, 8:20 am PST

8 Let J denote the Jacobian of a transformation $z \mapsto$ w. Referring to the previous question, show that the determinant is given by det(J) = $|\,\partial_z w\,|^2 - |\,\partial_{\bar z} w\,|^2$.

In my first attempt at this problem, I wrote the determinant of the Jacobian of w = u + iv on the axes z and $\bar{z}$ as

$$\det(J(w)) = \begin{vmatrix} \partial_z u & \partial_{\bar z} u \\ \partial_z v & \partial_{\bar z} v \end{vmatrix} \qquad \text{(see p. 193)}$$

where the vertical brackets signify the determinant, not the absolute value. If I assumed w to be analytic, this led to the correct answer, but Vasco pointed out that the Jacobian here is not the Jacobian of z $\mapsto$ w. It is something else, call it Z $\mapsto$ w.

Use instead the determinant from p. 193 unaltered and write the Jacobian for z = x + iy.

$$\det J(w) = \begin{vmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{vmatrix} = \partial_x u\, \partial_y v - \partial_y u\, \partial_x v$$

$$\partial_z = \tfrac{1}{2}\,\overline{\nabla}, \qquad \partial_{\bar z} = \tfrac{1}{2}\,\nabla$$

Show that $\det J(w) = |\,\tfrac{1}{2}\,\overline{\nabla} w\,|^2 - |\,\tfrac{1}{2}\,\nabla w\,|^2$

$$\overline{\nabla} w = \left(\partial_x - i\,\partial_y\right)(u + \text{iv})$$

$$= (\partial_x\text{u} + \partial_y\text{v}) + \text{i}\,(\partial_x\text{v} - \partial_y\text{u})$$

$$|\,\tfrac{1}{2}\,\overline{\nabla} w\,|^2 = \tfrac{1}{4}[\,\left(\partial_x u + \partial_y v\right)^2 + \left(\partial_x v - \partial_y u\right)^2]$$

$$= \tfrac{1}{4}[\,(\partial_x u)^2 + \left(\partial_y v\right)^2 + 2\partial_x u \partial_y v + (\partial_x v)^2 + \left(-\partial_y u\right)^2 - 2\partial_x v \partial_y u]$$

$$= \tfrac{1}{4}[\,(\partial_x u)^2 + (\partial_x v)^2 + \left(\partial_y v\right)^2 + \left(\partial_y u\right)^2 + 2\,(\partial_x u \partial_y v - \partial_x v \partial_y u)]$$

$$\nabla \text{w} = \left(\partial_x + i\,\partial_y\right)\,(u + \text{iv})$$

$$= (\partial_x\text{u} - \partial_y v) + \text{i}\,(\partial_x\text{v} + \partial_y\text{u})$$

$$|\,\tfrac{1}{2}\,\nabla w\,|^2 = \tfrac{1}{4}\left[\left(\partial_x u - \partial_y v\right)^2 + \left(\partial_x v + \partial_y u\right)^2\right]$$

$$= \frac{1}{4}[\ (\partial_x u)^2 + (\partial_x v)^2 + \left(\partial_y v\right)^2 + \left(\partial_y u\right)^2 + 2\ (-\partial_x u \partial_y v + \partial_x v \partial_y u)]$$

Then

$$\left|\ \frac{1}{2}\ \overline{\nabla} w\ \right|^2 - \left|\ \frac{1}{2}\ \nabla w\ \right|^2 = \frac{1}{4}[2\ (\partial_x u \partial_y v - \partial_x v \partial_y u)\ -2\ (-\partial_x u \partial_y v + \partial_x v \partial_y u)]$$

$$= \partial_x u \partial_y v - \partial_x v \partial_y u$$

Then $\left|\ \frac{1}{2}\ \overline{\nabla} w\ \right|^2 - \left|\ \frac{1}{2}\ \nabla w\ \right|^2 = \det J(w)$. This is what we wanted.

But this solution doesn't make the best use of the nabla notation with which one can obtain the same results with far fewer symbols, largely avoiding the cumbersome and easily confused subscripted partial notation, but admittedly introducing symbols that are less easily typed. The main advantages come in improved legibility and economies of space. Vasco has provided a solution that I attempt to replicate here just for the practice.

Retain the above calculation of $\det J(w) = \partial_x u\ \partial_y v - \partial_y u\ \partial_x v$ and show that $\det J(w) = \left|\ \frac{1}{2}\ \overline{\nabla} w\ \right|^2 - \left|\ \frac{1}{2}\ \nabla w\ \right|^2$. Let $w = u + iv$, where u and v are real functions of x and y.

$$\overline{\nabla} w = \overline{\nabla} u + i \overline{\nabla} v$$
$$\nabla w = \nabla u + i \nabla v$$

$$\left|\ \frac{1}{2}\ \overline{\nabla} w\ \right|^2 = \frac{1}{4}[\left(\overline{\nabla} u + i \overline{\nabla} v\right)(\nabla u - i \nabla v)]$$

$$= \frac{1}{4}\left[\ |\nabla u|^2 + \ |\nabla v|^2 + i(\overline{\nabla} v \nabla u - \overline{\nabla} u \nabla v)\right]$$

$$\left|\ \frac{1}{2}\ \nabla w\ \right|^2 = \frac{1}{4}[(\nabla u + i \nabla v)\left(\overline{\nabla}\ u - i\ \overline{\nabla} v\right)]$$

$$= \frac{1}{4}\left[\ |\nabla u|^2 + \ |\nabla v|^2\ +\ i\ (\nabla v\ \overline{\nabla} u - \ \nabla u \overline{\nabla} v)\ \right]$$

Then

$$\left|\ \frac{1}{2}\ \overline{\nabla} w\ \right|^2 - \left|\ \frac{1}{2}\ \nabla w\ \right|^2 = \frac{i}{4}(2 \overline{\nabla} v \nabla u\ - 2\ \nabla v\ \overline{\nabla} u)$$

$$= \frac{i}{2}\ \left(\overline{\nabla} v\ \nabla u - \overline{\overline{\nabla} v\ \nabla u}\right)$$

$$= \frac{i}{2}(2\ i\ \mathrm{Im}\ (\overline{\nabla} v\ \nabla u)) = -\ \mathrm{Im}\ (\overline{\nabla} v\ \nabla u)$$

$$= -\ \mathrm{Im}[\ (\partial_x v\ -\ i\ \partial_y v)(\partial_x u\ +\ i\ \partial_y u)]$$

$$= -\operatorname{Im}(\partial_x u\, \partial_x v + i\, \partial_x v \partial_y u - \mathrm{i}\partial_x u\, \partial_y v + \partial_y u \partial_y v)$$

$$= \partial_x u\, \partial_y v - \partial_x v \partial_y u = \det \mathrm{J}(\mathrm{w})$$

This is what we wanted.