Needham, Chapter 12, Exercise 7, p. 570
G. Palmer, January 21, 2019, 3:10 pm PST

7 (i) By thinking of z and $\overline{z}$ as functions of x and y, use the chain rule of partial differentiation to show (at least formally) that

$$\partial_z = \tfrac{1}{2}\,\overline{\nabla} \qquad \textit{and} \qquad \partial_{\overline{z}} = \tfrac{1}{2}\,\nabla$$

Apply the chain rule with ∇ or ∂, or with both? I was lost on this problem until I peeked at Vasco's answer and saw that z and $\overline{z}$ are not defined as **any** functions of x and y, but as the usual z = x + iy and $\overline{z}$ = x - iy. However, I didn't read far enough to discover that he had taken a different direction. But this is my attempt. It seemed to work for the first equation, but not for the second. I followed a variation of his approach on the second equation and later returned to the first equation and added it in.

Show $\partial_z = \frac{1}{2}\,\overline{\nabla}$. I am assuming that ∂_z is shorthand for ∂_zf and $\overline{\nabla}$ is shorthand for $\overline{\nabla}$f.

$$\partial_z(\text{f}[\text{z}]) = (\partial_z f)\,\partial_z z = \partial_z f\,\cdot 1 = \text{f}'$$

and

$$\overline{\nabla}\text{f}[\text{z}] = \text{f}'[\text{z}]\overline{\nabla}\text{z} \qquad \text{[from p. 518, bullet 3]}$$

$$= \text{f}'[\text{z}]\ 2\text{z}' = 2\ \text{f}'\cdot 1 = 2\ \text{f}'$$

We find that $\partial_z = \frac{1}{2}\,\overline{\nabla}$. QED.

The second approach is to derive ∂_z using the chain rule applied to f = u(x, y) + iv(x, y). Normally, we would write z = x + iy and $\overline{z}$ = x - iy. Vasco has suggested that Needham is actually asking us to reverse the focus to write x = $\frac{1}{2}$(z + $\overline{z}$) and y = $\frac{1}{2i}$(z - $\overline{z}$). These equations are then used to derive $\partial_z x$, $\partial_z y$, $\partial_{\overline{z}} x$, and $\partial_{\overline{z}} y$, which are used in substitutions into the chain.

Show that $\partial_z = \frac{1}{2}\overline{\nabla}$f.

Let f = u(x, y) + iv(x, y)

Let z = x + iy

Find $\partial_z f$ using chain rule.

$$\partial_z f = \partial_z u(x, y)\ +\ i\,\partial_z v(x, y)$$

$$= \partial_x u\partial_z x + \partial_y u\partial_z y + \text{i}(\partial_x v\partial_z x + \partial_y v\partial_z y) \qquad (1)$$

Find $\partial_z x = \frac{1}{2}, \partial_z y = \frac{1}{2i}$ using x = $\frac{1}{2}(z + \bar{z})$, y = $\frac{1}{2i}(z + \bar{z})$.

$$\partial_z x = \partial_z\left[\frac{1}{2}\,(z+\bar{z})\right] = \frac{1}{2}(\partial_z z + \partial_z \bar{z}) = \frac{1}{2}(1 + 0) = \frac{1}{2}$$

$$\partial_z y = \partial_z\left[\frac{1}{2i}\,(z-\bar{z})\right] = \frac{-i}{2}(\partial_z z - \partial_z \bar{z}) = \frac{-i}{2}(1 + 0) = \frac{-i}{2}$$

Find $\partial_z f$ using substitutions for $\partial_z x$, $\partial_z y$.

$$\partial_z f = \partial_x u \frac{1}{2} + \partial_y u \frac{-i}{2} + \mathrm{i}(\partial_x v \frac{1}{2} + \partial_y v \frac{-i}{2})$$

$$= \frac{1}{2}(\,\partial_x u - \mathrm{i}\partial_y u + \mathrm{i}\partial_x v + \partial_y v)$$

$$= \frac{1}{2}[(\,\partial_x - \mathrm{i}\partial_y)\,u + (\mathrm{i}\partial_x + \partial_y)\,v]$$

$$= \frac{1}{2}[(\,\partial_x - \mathrm{i}\partial_y)\,u + \mathrm{i}\,(\partial_x - i\,\partial_y)\,v]$$

$$= \frac{1}{2}\overline{\nabla}\,(\mathrm{u} + \mathrm{i}\ \mathrm{v}) = \frac{1}{2}\overline{\nabla} f$$

We can write this in the short form $\partial_z = \frac{1}{2}\overline{\nabla}$.

QED

Vasco's approach.

Show $\partial_{\bar{z}} = \frac{1}{2}\,\nabla$ (the $\bar{z}$ subscript is conjugate z).

Vasco (personal communication, Forum) has pointed out that it is not necessary to use f = u(x, y) + iv(x,y). Applying the chain rule from outer to inner functions, one could write

$$\partial_z f(x, y) = \partial_x \mathrm{f} \cdot \partial_z x + \partial_y \mathrm{f} \cdot \partial_z y$$

$$= (\partial_z x\ \cdot \partial_x + \partial_z y\ \cdot \partial_y)\ \mathrm{f}$$

which can be written as the operator $\partial_z x\ \cdot \partial_x + \partial_z y\ \cdot \partial_y$. Then one has

$$\partial_z = \partial_z x \cdot \partial_x + \ \partial_z y \cdot \partial_y \quad \text{and} \quad \partial_{\bar{z}} = \partial_{\bar{z}} x \cdot \partial_x + \partial_{\bar{z}} y \cdot \partial_y \quad \text{(1 a and b)}$$

The Wirtinger derivatives appear when we substitute into (1 a and b) using the derivations of ∂_zx, ∂_zy, $\partial_{\bar{z}}$x, and $\partial_{\bar{z}}$y from above.

Vasco is then able to conclude that

$$\partial_z = \frac{1}{2}\left(\partial_x - i\,\partial_y\right) = \frac{1}{2}\overline{\nabla}$$

The result was obtained by substituting $\frac{1}{2}$ and $\frac{-i}{2}$ into ∂_zx and ∂_zy in (1a).

$$\partial_z = \frac{1}{2}\,\partial_x + \frac{-i}{2}\,\partial_y = \frac{1}{2}\left(\partial_x - i\,\partial_y\right)$$

Then, by substitution for $\partial_{\overline{z}}$x and $\partial_{\overline{z}}$y into (1b),

$$\partial_{\overline{z}} = \frac{1}{2}\left(\partial_x + i\,\partial_y\right) = \frac{1}{2}\nabla$$

It appears that we have derived the Wirtinger derivatives and set them equal to our gradients times (1/2), or we have found that our gradient operators are handy symbols for the Wirtinger derivatives <https://en.wikipedia.org/wiki/Wirtinger_derivatives#Formal_definition>.

$$\partial_z = \frac{1}{2}(\partial_x - i\,\partial_y) = \frac{1}{2}\overline{\nabla}$$

$$\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\,\partial_y) = \frac{1}{2}\,\nabla\,(\partial_{\overline{z}} \text{ has conjugate(z))}$$

This satisfies part (i). Vasco's derivation appears to be the one that best suits the problem, as Needham made no mention of using f = u + iv, and because it works for both ∂_z and $\partial_{\overline{z}}$, and it is much more economical.

(ii) Deduce (at least formally) that an analytic function f depends on z but not on $\overline{z}$.

$$\partial_z f \;=\; f\,'\ \textit{and}\ \partial_{\overline{z}} f = 0$$

$$\partial_z \text{f} = \frac{1}{2}(\partial_x - i\,\partial_y)\,\text{f} = \frac{1}{2}\overline{\nabla}\text{f} = \frac{1}{2}2\text{f}' = \text{f}' \qquad \text{(from p. 518)}$$

$$\partial_{\overline{z}} f = \frac{1}{2}(\partial_x + i\,\partial_y)\,\text{f} = \frac{1}{2}\,\nabla\text{f} = \frac{1}{2}\cdot 0 = 0 \qquad \text{(from p. 518)}$$

Since $\partial_z f$ equals f', it depends on f(z), so f depends on z. We can also reason by (11, p. 483) that f = $\int_K f\,'\,dz = \int_K\left(W\left[\overline{f\,'},\ K\right] + \text{iF}\left[\overline{f\,'}, K\right]\right) dz$, where K is a contour, so f depends on z as it traverses K. Since $\partial_{\overline{z}} f = 0$, there is no change of f on $\overline{z}$. Then by definition, f does not depend on $\overline{z}$.