Needham, Chapter 12, Exercise 6, p. 570.
G. Palmer, January 16, 2019, 9:15 am PST

6 Let f(z) be analytic. By applying $\overline{\nabla}$ in turn to $(f + \overline{f})$ and to $(f\overline{f})$, show that if either the real part or the modulus of f is constant, then f itself is constant.

Let f = u + iv, where u and v are both real. Let u = const.

$$\overline{\nabla}(\text{const.}) \equiv \begin{pmatrix} \partial_x \\ -\partial_y \end{pmatrix} \cdot \text{const.} = \begin{pmatrix} \partial_x \text{const} \\ -\partial_y \text{const} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \equiv 0$$

$\overline{\nabla}(\mathrm{f} + \overline{f}) = 2\mathrm{f}'$ because $\overline{\nabla}\mathrm{f} = 2\mathrm{f}'$ and $\overline{\nabla}\overline{f} = \overline{(\nabla f)} = \overline{0} = 0$ (p. 518)

Since $(\mathrm{f} + \overline{f}) = 2\,\mathrm{Re(f)} = 2\mathrm{u}$, and $\overline{\nabla}(\text{const.}) = 0$, then f' = 0 and f must be a constant.

In the second part, I attempted to continue using the partial derivatives, but was unable to obtain a solution. The following follows Vasco's approach of expanding $\nabla(\mathrm{f}\,\overline{f})$.

If |f| = const., then $|f|^2$ = const. and $\overline{\nabla}\,|f|^2 = 0$ because both the partials are 0 [see first part],

$$\overline{\nabla}\,|f|^2 = \overline{\nabla}(f\overline{f}) = \left(\overline{\nabla} f\right)\overline{f} + \mathrm{f}\,\overline{\nabla}\overline{f}$$

$$= \left(\overline{\nabla} f\right)\overline{f} \qquad \text{(because } \overline{\nabla}\overline{f} = 0\text{, from first part)}$$

$$= 2\mathrm{f}'\,\overline{f} = 0$$

Then either $\overline{f} = 0$, f' = 0, or both are true. The first implies that f = 0, which is a constant, but not a general solution. The second implies that f is a constant, any constant. So f is also a constant.