Needham, Chapter 12, Exercise 5, p. 570.
G. Palmer, January 14, 2019, 10:25 am PST

5 (i) If f is analytic, show that $\nabla|f| = (f\overline{f'})/|f|$*, and explain how this agrees with part (i) of Ex. 19, p. 374.*

See page 523, top for a parallel derivation of $\nabla|f'|$.

$$2|f|\nabla|f| = \nabla\,|f|^2 = \nabla\,(f\overline{f}) = f\nabla\overline{f} + \overline{f}\,\nabla f = 2f\overline{f'}$$

$$\Longrightarrow \nabla|f| = \frac{f\overline{f'}}{|f|}$$

Proving step 4. a. Note that $\overline{f}\,\nabla f = 0$

$$f\nabla\overline{f} = f(\overline{\overline{\nabla} f}) = \mathrm{f}(\overline{2f'}) = 2\mathrm{f}\overline{f'}$$

Proving step 4. b. Brute force. Let f = u + iv.

$$\mathrm{f}\nabla\overline{f} = f\left(\partial_x + i\,\partial_y\right)(\mathrm{u - iv})$$

$$= \mathrm{f}\cdot[(\partial_x u - \mathrm{i}\partial_x v) + (\partial_y \mathrm{v} + \mathrm{i}\partial_y \mathrm{u})]$$

$$= \mathrm{f}\,(\overline{f'} + \overline{f'}) = 2f\overline{f'} \qquad \text{(from p. 210, (7), (8))}$$

Explain how this agrees with part (i) of Ex. 19, p. 374 (which has “the modulus of f increases most rapidly in the direction [f(p)/f’(p)]).

Our result is $\nabla|f| = \frac{f\overline{f'}}{|f|}$, which has the direction of $f\,\overline{f'}$. If $f = re^{i\phi}$, and $\overline{f'} = \left|f'\right|e^{-i\gamma}$, where γ = arg(f’), then arg($f\,\overline{f'}$) = ϕ - γ = arg(f/f’). Hence, $\nabla|f| = \frac{f\overline{f'}}{|f|}$ has the same direction as that in which the modulus of f increases most rapidly.

(ii) Show that if Δ *is the Laplacian, then* $\Delta\,|f|^2 = 4|f'|^2$*. Try deriving this by brute force.*

The less brutish derivation:

$$\Delta\,|f|^2 = \overline{\nabla}\nabla|f|^{\,2} \qquad \text{(by p. 519, bullet)}$$

$$= \overline{\nabla}\,(2f\overline{f'}) = 2\,[\,\overline{f'}\left(\overline{\nabla}f\right) + \mathrm{f}\,\overline{\nabla}\overline{f'}] \qquad \text{(see part i and p. 524, bot., } \nabla f' = 0 = \nabla\overline{f'}\text{)}$$

$$= 2\ \overline{f'}\,(2 f') = 4\ |f'|^2$$

Try deriving this by brute force.

True brute force might mean

$$\Delta |f|^2 = (\partial_x^2 + \partial_y^2)(u^2 + v^2)$$

$$= \begin{pmatrix}\partial_x \\ \partial_y\end{pmatrix} \cdot \begin{pmatrix}\partial_x \\ \partial_y\end{pmatrix} \left(u^2 + v^2\right)$$

$$= \begin{pmatrix}\partial_x \\ \partial_y\end{pmatrix} \cdot \begin{pmatrix}2\,(u\,\partial_x u\ +\ v\,\partial_x v) \\ 2\left(u\,\partial_y u\ +\ v\,\partial_y v\right)\end{pmatrix}$$

$$= 2\,[\partial_x(\mathrm{u}\,\partial_x u\ +\ v\,\partial_x v) + \partial_y\left(u\,\partial_y u\ +\ v\,\partial_y v\right)]$$

$$= 2[(\partial_x u)^2 + \mathrm{u}\,\partial_x^2\mathrm{u} + (\partial_x v)^2 + \mathrm{v}\,\partial_x^2\mathrm{v} + \left(\partial_y u\right)^2 + \mathrm{u}\,\partial_y^2\mathrm{u} + \left(\partial_y v\right)^2 + \mathrm{v}\,\partial_y^2\mathrm{v}]$$

$$= 2\left[\left((\partial_x u)^2\ + (\partial_x v)^2\right) + \ \left(\left(\partial_y u\right)^2 +\ \left(\partial_y v\right)^2\right)\right] \quad \text{[because u } \left(\partial_x^2\mathrm{u} + \partial_y^2\mathrm{u}\right) = 0\text{, also w. v]}$$

$$= 2(|f'|^2 + |f'|^2) \qquad \text{[using (7) and (8), p. 210]}$$

$$= 4\ |f'|^2$$

QED.

One could also write (given that $\partial_x^2 u^2 = 2\left((\partial_x u)^2 + \mathrm{u}\partial_x^2\right)$)

$$\Delta |f|^2 = (\partial_x^2 + \partial_y^2)(u^2 + v^2)$$

$$= 2\left[(\partial_x u)^2 + \mathrm{u}\partial_x^2\mathrm{u} + (\partial_x v)^2 + \mathrm{v}\partial_x^2\mathrm{v} + \left(\partial_y u\right)^2 + \mathrm{u}\partial_y^2\mathrm{u} + \left(\partial_y v\right)^2 + \mathrm{v}\partial_y^2 v\right] \quad \text{(cf. line 5, previous)}$$

$$= 2\left[\left((\partial_x u)^2\ + (\partial_x v)^2\right) + \ \left(\left(\partial_y u\right)^2 +\ \left(\partial_y v\right)^2\right)\right] \quad \text{[because u } \left(\partial_x^2\mathrm{u} + \partial_y^2\mathrm{u}\right) = 0\text{, also w. v]}$$

$$= 2(|f'|^2 + |f'|^2) \qquad \text{[using (7) and (8), p. 210]}$$

$$= 4\ |f'|^2$$