Needham, Chapter 12, Exercise 4, p. 570.
G. Palmer, January 21, 2019, 2:10 pm PST

4 Find the generalization of (7) in three dimensional space.

Let V = ∇Φ be the vector field of the potential function Φ. The flux of V out of a (non-infinitesimal) sphere S of radius r is then.

$$F[V, S] = \pi r^2\, \partial_r < \Phi >$$

We use a model similar to that of [18], p. 496, adapted to three dimensions. The lines of flux are rays and the sphere S forms a surface of constant potential. We draw a radius on the x axis. Then N = $(1, 0, 0)^{\text{transposed}}$ and r = x. The total flux of V out of S is the same no matter the radius of S, so the total is the sum of the fluxes at all points on the surface of S.

If V is sourceless, then F = 0 (by p. 517), and since $\partial_r < \Phi > = 0$, the average potential density <Φ> is independent of the radius of S. Shrinking S down to p, we deduce that this radius independent value must be Φ(p) (p. 517).

$$F[V, C] = \int_{\mathcal{S}} V \bullet N\, ds,$$

where $\mathcal{S}$ is the surface area of the sphere S. The infinitesimal ds is equivalent to dA in a general area integral.

Since the calculated value of V = $\nabla\Phi$ varies over $\mathcal{S}$, we take the average value <Φ> over this area. We treat <V•N> as a constant and let $\int_{\mathcal{S}} ds$ equal the basic geometric value of $4\pi r^2$ for the area of the surface of a sphere.

$$F[V, C] = 4\pi r^2 <V \bullet N> = 4\pi r^2 < \nabla\Phi \bullet N >$$

$$= 4\pi r^2 < \begin{pmatrix} \partial_x \Phi \\ \partial_y \Phi \\ \partial_z \Phi \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} >$$

$$= 4\pi r^2\, \partial_r < \Phi > \qquad (1)$$

Now suppose that V is not sourceless, but that its flux density $\nabla \bullet V = \Delta\Phi$ is constant. Gauss' Divergence Theorem [(7), p. 481] then yields $F[V, C] = (4/3)\,\pi r^3\, \Delta\Phi$. Inserting this into the previous result, we find that

$$\partial_r < \Phi > = \frac{1}{6} r^2\, \Delta\Phi$$

$\iint\int_S [\nabla \bullet \mathrm{V}]\, d\mathrm{Vol}$ [apply Gauss' divergence theorem]

$= \Delta\Phi \iint\int_S d\mathrm{Vol}$ [because $\nabla \bullet \mathrm{V} = \Delta\Phi$ is constant]

$= (4/3)\, \pi \mathrm{r}^3\, \Delta\Phi$ (2)

$\mathrm{F}[\mathrm{V}, \mathrm{C}] = 4\, \pi \mathrm{r}^2\, \partial_r < \Phi > = (4/3)\, \pi \mathrm{r}^3\, \Delta\Phi$ [(1) = (2)]

$\Longrightarrow \partial_r < \Phi > = \frac{(4/3)\, \pi \mathrm{r}^3\, \Delta\Phi}{4\, \pi \mathrm{r}^2} = \frac{r\, \Delta\Phi}{3}$

$\partial_r < \Phi > = \frac{1}{6}\, r^2\, \Delta\Phi$

Which may be integrated to yield the formula similar to that in (7), but adapted for three dimensions.

$\int \partial_r < \Phi > dr = \int \frac{r\, \Delta\Phi}{3}\, dr + \text{const.}$

$<\Phi> = \frac{1}{6}\, r^2\, \Delta\Phi + \Phi(\mathrm{p})$

$<\Phi> - \Phi(\mathrm{p}) = \frac{1}{6}\, r^2\, \Delta\Phi$