Needham, Chapter 12, Exercise 3, p. 570.
G. Palmer, December 20, 2018, 9 am PST

3 Use Gauss' Mean Value Theorem to show that a harmonic function cannot have a local maximum.

Defining a "local maximum". On p. 356, the term "local maximum" appears to refer to the value of a modular function $|f|$ at interior points only. "A modular surface has no [interior] peaks". Vasco (personal communication, Forum) pointed out that "A point can only be a local maximum if it is possible to draw a circle of infinitesimal radius centred at the point." Hence, a point on the boundary can not be a local maximum.

Gauss' mean value theorem states that *The average value of a harmonic function on a circle is equal to the value of the function at the centre of the circle* (p. 511).

Vasco has a proof by contradiction, which I attempt to reproduce here. Since complex functions do not have a maximum, the proof is limited to real harmonic functions of z (which might include modular functions of a complex function f, as on p. 356, but only if f is a constant). Let Φ be a harmonic function with a local maximum $\Phi(p)$. Since Φ is harmonic, by Gauss' mean value theorem $<\Phi> = \Phi(p)$, where $<\Phi>$ is the average over any circle centred at p. Since $\Phi(p)$ is a local maximum, points $\hat{p} \neq p$ within or on an infinitesimal circle centred at p have $<\Phi>$ less than $\Phi(p)$, but this contradicts the mean value theorem. Hence, a real harmonic function cannot have a local maximum.