Needham, Chapter 12, Exercise 21, p. 572.
G. Palmer, December 14, 2019, 9:05 am PST

21 Use the idea behind the method of images to show that if 0 < Re(p) < 1, then the Green's function of the half-disc Re(z) ≥ 0, |z| ≤ 1, is

$$\mathcal{G}_p(z) = -ln\left|\frac{z-p}{\overline{p}z-1}\right| + ln\left|\frac{z+\overline{p}}{pz+1}\right|$$

Check this by getting the computer to draw the level curves of $\mathcal{G}_p(z)$.

Older versions of VCA show this problem:

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Begin with the complex potential $\tilde{\Omega}(w)$ of a vortex on the unit disc [(21), p. 551]. Then

$$\tilde{Q}(w) = \text{i log w}$$

We are familiar with the first part $-ln\left|\frac{z-p}{\overline{p}z-1}\right|$ as the Green's function for a source of strength 2π that was obtained as the dual of a vortex located at p on the disc (p. 551).

$$\mathcal{G}(\text{z}) = -\,\phi(\text{z}) = -\,\ln\left|\frac{z-p}{\overline{p}z-1}\right| \qquad (21)$$

This vortex was found using the method of images by reflecting an undisturbed vortex in the unit circle (p. 549). The resulting "disturbed" vortex is (19):

$$\Omega(\text{z}) = \text{i log}\left[\frac{z-p}{\overline{p}z-1}\right] - \delta \qquad (19)$$

and the dual of that flow is a **source** of the same strength at p with the boundary at zero potential.

$$\Omega(\text{z}) = \text{log}\left[\frac{z-p}{\overline{p}z-1}\right] + \text{i}\delta$$

Reflect the **vortex** Ω(z) from (19) about the y-axis to obtain a formula for a **vortex** of equal and opposite strength on the right half-disc. This can be done by letting $f^{\ddagger}$(z) =

$\overline{f\,(-z)}$ using the formula with Schwartzian reflection $f^{\ddagger}(\mathrm{z}) = \overline{f[\mathcal{R}_K\,(z)]}$ from p. 538. Then let f = $\frac{z-p}{\overline{p}\,z-1}$, $f^{\ddagger}(\mathrm{z}) = -\frac{z+\tilde{p}}{p\,z+1}$, superpose this onto (19), and name the new constant γ.

$$\Omega_d(\mathrm{z}) = \mathrm{i}\,\log\left[\frac{z-p}{\overline{p}\,z-1}\right] - i\,\log\left[\frac{z+\tilde{p}}{p\,z+1}\right] - \gamma$$

For points of correspondence at which to measure the vanishing complex potentials, choose a boundary point $\tilde{a}$ on the y-axis of $\tilde{\Omega}$(w) to correspond to a on the right hemisphere of Ω(z) of the vortex in the disc. I believe that a = $\tilde{a}$ = i would work. Write $\tilde{\Omega}$(w) = Ω(z).

Find the dual of this composite vortex as a **source** of equal strength centred at p with zero flow on the boundary (y-axis; cf. [19] on x-axis). Note the duals $\hat{\Phi} = +\Psi$ and $\hat{\Psi} = -\Phi$, so that $\hat{\Omega} = \hat{\Phi} + \mathrm{i}\hat{\Psi} = -\,\mathrm{i}\,(\Phi + \mathrm{i}\Psi)$.

$$\Omega_{d\ \mathrm{src}}(\mathrm{z}) = \log\left[\frac{z-p}{\overline{p}\,z-1}\right] - \log\left[\frac{z+\overline{p}}{p\,z+1}\right] + \mathrm{i}\gamma$$

Then $\mathcal{G}_p(z) = -\Phi_{(\mathrm{src\ at\ }p)}(z) - \Phi_{(\mathrm{src\ at\ }-\overline{p})}(z)$.

$$\mathcal{G}_p(z) = -\ln\left|\frac{z-p}{\overline{p}\,z-1}\right| + \ln\left|\frac{z+\overline{p}}{p\,z+1}\right|$$

Check this by getting the computer to draw the level curves of $\mathcal{G}_p(z)$.

Out[413]=

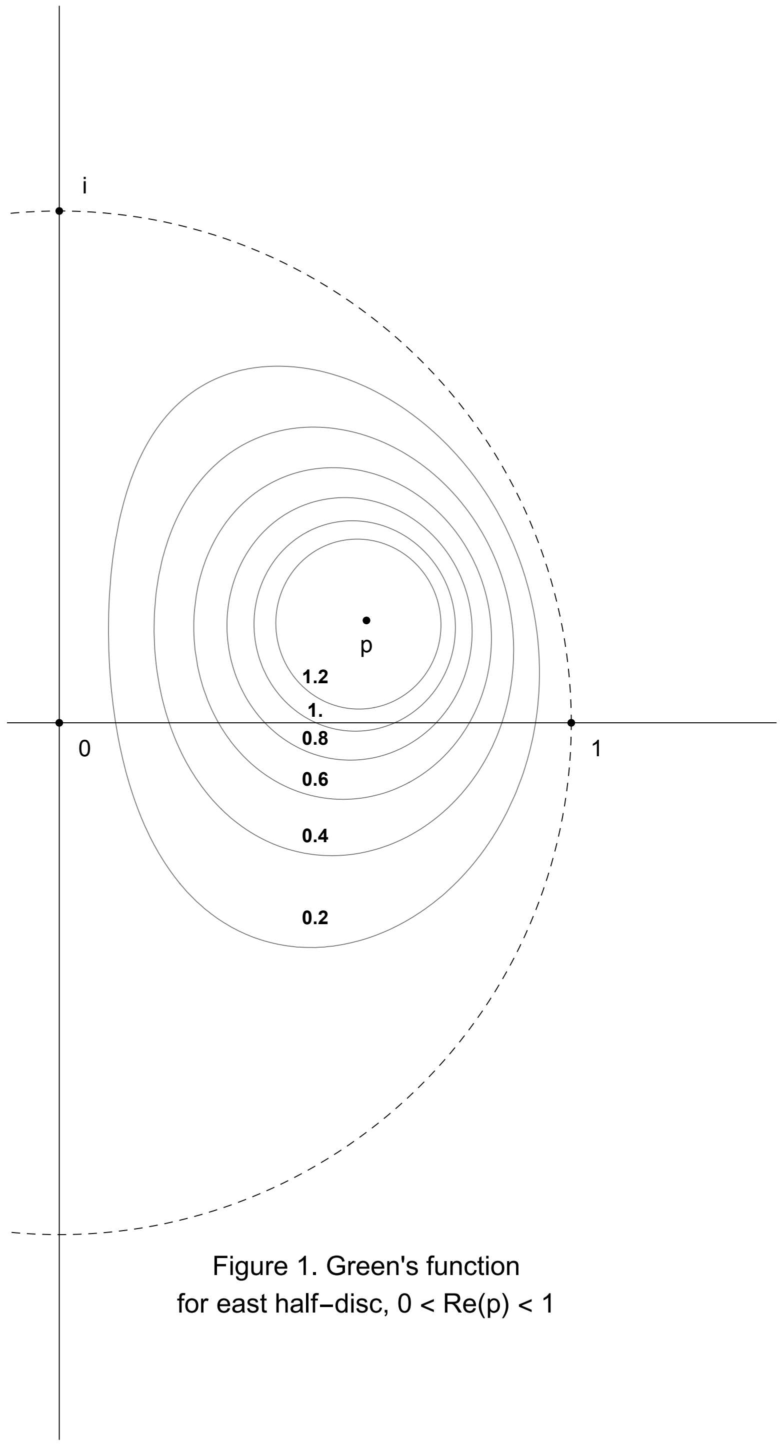

Figure 1. Green's function
for east half−disc, 0 < Re(p) < 1