**Needham, Chapter 12, Exercise 20, p. 572.**
**G. Palmer, December 7, 2019, 9:20 am PST**

*20 (i) Use the method of images to find the Green's function for the upper half plane.*

To use the method of images, we recognize that Green's function for the unit disc is based on a complex potential of an inverted vortex on the disc (pp. 549-551). We want the complex potential of a vortex on the unit disc reflected in the real line to the UHP. With a vortex of strength $2\pi$ centred at point $\overline{p}$ on the disc, we reflect it in the real line, which maps it to another vortex of equal and opposite strength centred at p in the upper half-plane. Reflection in the real line is accomplished with $f^*$:

$$f^*(z) = \overline{f(\overline{z})}$$

Let f(z) = $\Omega_u(z)$ = i log(z-$\overline{p}$) and write the superposition of the undisturbed vortex and its reflection:

$$\Omega_d(z) = \text{i}\log(z-\overline{p}) - \text{i}\log(z-p) - \gamma$$

$$= \text{i}\log\left[\frac{z-\overline{p}}{z-p}\right] - \delta$$

To ensure a mapping of equal potentials on the boundaries, on each vortex we choose a boundary point where circulation and flux vanish. We choose $\tilde{a} = 1$ and a = $\infty$ on the real line. This a is generally not the same point as **a** in (37) and (34). Solve for w:

$$\tilde{\Omega}(w) = \Omega(z) = \text{i}\log w = \text{i}\log\left[\frac{z-\overline{p}}{z-p}\right] - \delta$$

$$\log w = \log\left[\frac{z-\overline{p}}{z-p}\right] + \text{i}\delta$$

$$e^{\log w} = e^{\log\left[\frac{z-\overline{p}}{z-p}\right]+\text{i}\delta}$$

$$w = e^{\text{i}\delta}\left[\frac{z-\overline{p}}{z-p}\right]$$

Then, Green's function of the upper half-plane with pole at p is

$$\mathcal{G}_p(z) = -\ln|f(z)| = -\ln|w| = -\ln\left|\, e^{\text{i}\delta}\left[\frac{z-\overline{p}}{z-p}\right]\right| = -\ln\left|\frac{z-\overline{p}}{z-p}\right|$$

Note that $\mathcal{G}_p(z) = 0$ at z = $\infty$.

*(ii) Use this to show that Green's general formula (37) does indeed yield the Poisson half-plane formula (34).*

$$\mathrm{T}(\mathbf{a}) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{Y}{(X-x)^2+Y^2}\right]T(x)\,dx \qquad (34)$$

$$\mathrm{T}(\mathbf{a}) = \frac{1}{2\pi}\int_B \mathcal{Q}(z)\,T(z)\,ds \qquad (37)$$

On p. 565, Needham introduces Green's General Formula which finds T(a) "in terms of the values T(z) on the boundary B." Using Green's, it seems we begin with the distribution of T on the boundary of the reflected vortex and assign equal temperatures to the corresponding points of the x-axis, which is the boundary of the upper half-plane.

The distribution T(x) on the real line is given, so z must traverse the real line. This means that we must find a way to transform $\mathcal{Q}$ ds to $\frac{Y}{(X-x)^2+Y^2}$ dx. The values of the integrand depend on the distance of x from **a** at each point on the x axis and the given values of T(x) on the x-axis. At very large |x|, the influence of T(x) on T(a) decreases with increasing $|x|^2$. From part (i),

$$\mathcal{G}_p(x) = -\ln|f_d(\mathrm{x})| = -\ln\left|\frac{z-\overline{p}}{z-p}\right|$$

and from (25), p. 554, on the boundary the flux is

$$\mathcal{Q}(z) = |f'(z)|$$

Let f = $\frac{z-\overline{p}}{z-p}$. Then by the quotient rule,

$$f' = \frac{z-p-(z-\overline{p})}{(z-p)^2} = \frac{\overline{p}-p}{(z-p)^2}$$

Then, by (37),

$$\mathrm{T}(a) = \frac{1}{2\pi}\int_B \frac{|\overline{p}-p|}{|z-p|^2}\,T(z)\,ds$$

This **a** is generally not the same as the a at zero flux and potential in part (i). Using the format of (34), a = p = X + i Y, where Y ≥ 0.

$$| f'(z)| = \frac{|X-\mathrm{iY}-(X+\mathrm{iY})|}{(z-p)\,(\bar{z}-\bar{p})} = \frac{|-2\,\mathrm{iY})|}{|z|^2-\bar{p}\,z-p\,\bar{z}+|p|^2}$$

$$= \frac{2\,Y}{\left(x^2+y^2\right)-(X-\mathrm{iY})\,(x+\mathrm{iy})-(X+\mathrm{iY})\,(x-\mathrm{iy})+\left(X^2+Y^2\right)}$$

$$= \frac{2\,Y}{\left(x^2+y^2\right)-\mathrm{Xx}-\mathrm{iXy}+\mathrm{iYx}-\mathrm{Yy}-\mathrm{Xx}+\mathrm{iXy}-\mathrm{iYx}-\mathrm{Yy}}$$

$$= \frac{2\,Y}{\left(x^2+y^2\right)-2\,\mathrm{Xx}-2\,\mathrm{Yy}+\left(X^2+Y^2\right)}$$

The border is the real axis and (34) integrates on x, so let y = 0 and rewrite:

$$= \frac{2\,Y}{x^2-2\,\mathrm{Xx}+\left(X^2+Y^2\right)} = \frac{2\,Y}{(X-x)^2+Y^2}$$

Letting ds = dx, one can write

$$\mathrm{T(a)} = \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{Y}{(X-x)^2+Y^2}\right]T(x)\,dx$$

I think it helps to regard the real line as a circle of infinite radius. The point a (point of correspondence from part i) lies opposite the point 0 on this circle. As |**x**| approaches ∞, or as **a** (from part ii) approaches the point a of vanishing potential and flux (from part i), T(**a**) approaches 0.

Question: Does the point a of vanishing complex potential in R (the UHP) have the angle 0 (because it is on the real line), or $\pi/2$, because it is opposite 0 on the circle of infinite radius?

Comment: Paragraph 1, p. 565 introducing Green's general formula, says that Poisson's formula is for "finding the temperature at any point a inside R". He then derives it from Green's function "when the heat source is placed at point a". Thus, he seems to be saying that one can find the temperature of any point by introducing a heat source at that point, calculating Green's formula, calculating the flux on the boundary from Green's formula, and integrating flux times temperature on the boundary. In other words, if you know the temperature on the boundary, you can use Greens function to calculate flux on the boundary and find the temperature at any point. Integrate flux · temp on the boundary and you get T(a).