Needham, Chapter 12, Exercise 1, p. 505-506.
G. Palmer, December 4, 2018, 9:15 am PST

2 If Φ is a real harmonic function, and $\hat{\Phi}$ is its harmonic dual, show that $\Phi\hat{\Phi}$ is also harmonic. [Hint: Consider the complex function $(\Phi + i\hat{\Phi})$.]

Some relevant excerpts from Needham:

p. 258 real and imaginary parts of a analytic function are harmonic; *both automatically satisfy Laplace's equation:* $\Delta\,\Phi = 0$.

p. 511 "If we are given a real function $\Phi(z)$ then we may construct an irrotational vector field $\overline{H}$ for which Φ is the potential function, namely

$$\overline{H} = \nabla\,\Phi.$$

If Φ is harmonic then we know [(30), p. 500] that $\overline{H}$ will be sourceless, and so it will possess a stream function ψ. Since $\overline{H}$ is irrotational, Ψ is harmonic. **The complex potential $\boldsymbol{\Omega = \Phi + i\,\Psi}$ will then be an analytic function** [emphasis added] having as its real part the given harmonic function Φ. In other words, we have shown that

"The harmonic dual of a given harmonic function Φ is the stream function of the vector field $\nabla\,\Phi$."

p. 518 *A complex function is analytic if and only if $\overline{\nabla} f = 0$, in which case …*

The correct answer (forum message, Vasco):

By the hint, we are encouraged to consider the complex function $\Phi + i\hat{\Phi}$. From page 511, we deduce that both Φ and $\hat{\Phi}$ are harmonic, because $\hat{\Phi} = \Psi$, and that $\Phi + i\hat{\Phi}$ is analytic. This point will be important after squaring the function.

$$\left(\Phi + i\,\hat{\Phi}\right)^2 = (\Phi^2 - \hat{\Phi}^2) + i2\Phi\hat{\Phi}$$

Since the function $\Phi + i\,\hat{\Phi}$ is analytic, so is its square. Then $2\Phi\hat{\Phi}$ is the imaginary part of an analytic function, so it is harmonic by the statement from p. 258.

My answers, possibly correct.

<u>Solution with multiplication rule and Δ:</u> Since Δ is the product of ∇ and $\overline{\nabla}$ in either order, and since both ∇ and $\overline{\nabla}$ obey the multiplication rule (p. 518, bottom), we deduce that Δ also obeys the multiplication rule.

$$\Delta(\Phi\hat{\Phi}) = (\Delta\Phi)\,\hat{\Phi} + \Phi\,(\Delta\,\hat{\Phi})$$

But we know that both Φ and $\hat{\Phi}$ are harmonic, so $\Delta\Phi = \Delta\,\hat{\Phi} = 0$. Hence, $\Delta(\Phi\hat{\Phi}) = 0$, which is what we wanted.

Solution with partial derivatives: The solution makes heavy use of the differentiation of the product rule $\nabla(fg) = f\nabla g + g\nabla f$ (p. 518). The format of the partials wrt y are the same as for the partials wrt x, so we abbreviate them with "∂_y terms".

$$\Delta(\Phi\hat{\Phi}) = (\partial_x^2 + \partial_y^2)\,\Phi\hat{\Phi}$$

$$= \partial_x^2(\Phi\hat{\Phi}) + \partial_y^2(\Phi\hat{\Phi})$$

$$= \partial_x(\partial_x(\Phi\hat{\Phi})) + \partial_y(\partial_y(\Phi\hat{\Phi}))$$

$$= \partial_x(\partial_x\Phi \cdot \hat{\Phi} + \Phi \cdot \partial_x\hat{\Phi}) + \partial_y(\partial_y\Phi \cdot \hat{\Phi}) + \Phi \cdot \partial_y\hat{\Phi})$$

$$= (\partial_x(\partial_x\Phi) \cdot \hat{\Phi} + \partial_x\Phi \cdot \partial_x\hat{\Phi}) + (\,\partial_x\Phi \cdot \partial_x\hat{\Phi} + \Phi \cdot \partial_x(\partial_x\hat{\Phi})) + \partial_y \text{ terms}$$

$$= \partial_x^2\Phi \cdot \hat{\Phi} + \partial_x\Phi \cdot \partial_x\hat{\Phi} + \partial_x\Phi \cdot \partial_x\hat{\Phi} + \Phi \cdot \partial_x^2\hat{\Phi}\text{s} + \partial_y \text{ terms}$$

The terms containing 2nd derivatives are 0 because Φ and $\hat{\Phi}$ are harmonic. Now we add back in the terms with ∂_y.

$$\Delta(\Phi\hat{\Phi}) = \partial_x\Phi \cdot \partial_x\hat{\Phi} + \partial_x\Phi \cdot \partial_x\hat{\Phi} + \partial_y\Phi \cdot \partial_y\hat{\Phi} + \partial_y\Phi \cdot \partial_y\hat{\Phi}$$

Since $\Phi + i\hat{\Phi}$ is analytic, we can make Cauchy-Riemann substitutions, shown in red.

$$\Delta(\Phi\hat{\Phi}) = \partial_x\Phi \cdot \partial_x\hat{\Phi} + \partial_x\Phi \cdot \partial_x\hat{\Phi} - \partial_x\hat{\Phi} \cdot \partial_x\Phi - \partial_x\hat{\Phi} \cdot \partial_x\Phi$$

$$= 0$$

This is what we wanted.