Needham, Chapter 12, Exercise 19, p. 572.
G. Palmer, November 29, 2019, 12 pm

19 Substitute the disc Green's function (21) into (24), thereby confirming the formula (26) for the heat flux at the boundary of the disc.

$$\mathcal{G}_p(z) = -\Phi(z) = -\ln\left|\frac{z-p}{\overline{p}\,z-1}\right| \qquad (21)$$

$$\mathcal{Q}(z) = -\frac{\partial \mathcal{G}}{\partial n} \qquad (24)$$

$$\mathcal{Q}(z) = \frac{1-|p|^2}{\rho^2} \qquad (26)$$

This entire answer follows Vasco's answer very closely. From Vasco's answer to part (iii), Ex. 15, p. 4 beginning "Using (24) on p. 553 where n measures distance in the direction of N", we can write a slightly modified set of equations:

$$\mathcal{Q}(z) = -\frac{\partial \mathcal{G}}{\partial n} = |\,\nabla \mathcal{G}\,(z) \bullet N\,| = |\nabla \mathcal{G}\,(z)|, \text{ where N = z on the unit circle C}$$

Then, let f(z) = $\frac{z-p}{\overline{p}\,z-1}$.

$$\mathcal{Q}(z) = \left|\,\nabla\left(-\ln\left|\,e^{i\delta}\,\frac{z-p}{\overline{p}\,z-1}\,\right|\right)\right| = \left|\,\nabla\left(\ln\left|\,\frac{z-p}{\overline{p}\,z-1}\,\right|\right)\right|$$

$$\nabla\mathcal{G} = (\partial_x + i\,\partial_y)(\mathcal{G})$$

$$\partial_x\mathcal{G} = -\frac{1}{|f|}\,\partial_x\left|\,f\,\right|$$

$$\partial_y\mathcal{G} = -\frac{1}{|f|}\,\partial_y\left|\,f\,\right|$$

$$\partial_x\,|\,f\,|^2 = 2\,\;|\,f\,|\,\partial_x\,|\,f\,|$$

$$\partial_x\,|\,f\,| = \frac{\partial_x|f|^2}{2|f|} \qquad \text{by dividing through by 2|f|}$$

$$\partial_y\,|\,f\,| = \frac{\partial_y|f|^2}{2|f|}$$

$$\partial_x\mathcal{G} = -\frac{1}{|f|}\,\frac{\partial_x|f|^2}{2|f|} = -\frac{\partial_x|f|^2}{2|f|^2} \qquad (1)$$

$$\partial_y \mathcal{G} = -\frac{1}{|f|}\,\frac{\partial_y |f|^2}{2|f|} = -\frac{\partial_y |f|^2}{2|f|^2} \qquad (2)$$

Let A = $|z-p|^2$ and $B = |\overline{p}\,z - 1|^2$. The letters A and B are used in place of Vasco's U and V to avoid possible confusion with the u and v of a typical analytic function f = u + iv. We have shown that A = B (See the exercise following the problem.), so

$$|\ f\ |^2 = \frac{|z-p|^2}{|\overline{p}\,z-1|^2} = \frac{A}{B} = 1$$

Substitute 1 into $\left|f^2\right|$ in the denominator of (1) and (2), but not into $\partial_x\ |\ f\ |^2$. We retain $\left(\partial_x\ |\ f\ |^2\right)_{|z|=1}$ in the numerator because the value of the derivative with respect to x on the unit circle remains to be discovered.

$$\partial_x \mathcal{G} = -\frac{\partial_x |f|^2}{2} = -\frac{1}{2}\,\partial_x \frac{A}{B} = -\ \frac{(\partial_x A)\,B - A\,(\partial_x B)}{2\,B^2}$$

Since A = B = ρ^2,

$$\partial_x\ |\ f\ |^2 = \ \frac{(\partial_x A)\,B - A\,(\partial_x B)}{\rho^4} = \frac{(\partial_x A)\ -\ (\partial_x B)}{\rho^2}$$

$$\partial_y\ |\ f\ |^2 = \frac{(\partial_y A)\ -\ (\partial_y B)}{\rho^2}$$

Find ∂_x of A and B.

$$\partial_x A = \partial_x A = \partial_x\ |\ z - p\ |^2$$

$$= \partial_x\ [(\text{z-p})\ (\overline{z}\ \text{-}\ \overline{p})] = \ \partial_x\left(\ |\ z\ |^2\ \text{-}\ \overline{p}\text{z - p}\overline{z} +\ |\ p\ |^2\right)$$

$$= \partial_x\ |\ z\ |^2\ \text{-}\ \overline{p}\ \text{- p} = \partial_x\left(x^2 + y^2\right)\ \text{-}\ \overline{p}\ \text{- p = 2x - (}\overline{p}\text{ + p)}$$

$$\partial_x B = \partial_x\ \ |\ \overline{p}\,z - 1\ |^2$$

$$= \partial_x[(\overline{p}\,z - 1)\ (\text{pz} - 1)] = \partial_x\left(\ |\ p\ |^2\ \ |\ z\ |^2\ \text{-}\ \overline{p}\,z\ \text{- pz + 1}\right)$$

$$= \partial_x\left(\ |\ p\ |^2\ \ |\ z\ |^2\right)\ \text{-}\ \overline{p}\ \text{- p}\ ) = \partial_x\left(\left|\ \ p\ \ \right|^2\ \left(x^2 + y^2\right)\right)\ \text{-(}\ \overline{p} + \text{p)}$$

$$= 2\,x\,|\;p\;|^2 - (\,\overline{p} + \text{p}\,)$$

Then

$$\partial_x \mathcal{G} = -\,\frac{(\partial_x A)\;-\;(\partial_x B)}{2\,\rho^2} = -\,\frac{2\,x-(\overline{p}+p)\;-\left(2\,x|\,p\,|^2-(\overline{p}+p)\right)}{2\,\rho^2} = -\,\frac{x\left(1-|p|^2\right)}{\rho^2}$$

Find ∂_y of A and B.

$$\partial_y A = \partial_y\;|\;z - p\;|^2$$

$$= \partial_y\;[(z\text{-p})(\overline{z}\text{-}\overline{p})] = \partial_y\Big[\;|\;z\;|^2 - \overline{p}\text{z} - \text{p}\overline{z} + \;|\;p\;|^2\Big]$$

$$= \partial_y\left(x^2 + y^2\right) - \text{i}\overline{p} + \text{ip}$$

$$= 2\text{y} + \text{i}\;(\text{p-}\overline{p})$$

$$\partial_y B = \partial_y\;|\;\overline{p}\,z - 1\;|^2 = \partial_y[(\overline{p}\,z - 1)(p\,\overline{z} - 1)]$$

$$= \partial_y\Big[\;|\;p\;|^2\left(x^2 + y^2\right) - \overline{p}\,z - p\,\overline{z} + 1\Big]$$

$$= 2\text{y}\;\;|\;p\;|^2 - i\,\overline{p} + \text{ip} = 2\text{y}\;\;|\;p\;|^2 + \text{i}\;(\text{p-}\overline{p})$$

Then

$$\partial_y \mathcal{G} = -\,\frac{\left(\partial_y A\right)\;-\;\left(\partial_y B\right)}{2\,\rho^2} = -\,\frac{2\,y + i\,(p-\overline{p})\;-2\,y\,|\,p\,|^2 - i\,(p-\overline{p})}{2\,\rho^2} = -\,\frac{y\left(1-|p|^2\right)}{\rho^2}$$

By using $|\,\nabla\mathcal{G}\,|^2$ rather than $(\partial_x + i\,\partial_y)\,\mathcal{G}$, we avoid the canceling signs that would result in zero.

$$|\,\nabla\mathcal{G}\,|^2 = (\partial_x\mathcal{G})^2 + (\partial_y\mathcal{G})^2 = x^2\left[\frac{\left(1-|p|^2\right)}{\rho^2}\right]^2 + y^2\left[\frac{\left(1-|p|^2\right)}{\rho^2}\right]^2$$

$$= \left[\frac{\left(1-|p|^2\right)}{\rho^2}\right]^2, \text{ because } \left(x^2 + y^2\right) = |\text{z}| = 1 \text{ on C}$$

$$|\,\nabla\mathcal{G}\,| = \frac{1-|p|^2}{\rho^2} \text{ as was to be shown.}$$

Multiplying by N = $e^{i\theta}$ inside the absolute brackets does not change the result.

Exercise, p. 554: At the boundary point z, the local heat flux is equal to the amplification of f: $\mathcal{Q}(z) = |f'(z)|$. Using (20), the result predicts that the local heat flux at the boundary of the unit disc is given by [exercise].

$$\mathcal{Q}(z) = |f'(z)| = \frac{1-|p|^2}{|z-p|^2} = \frac{1-|p|^2}{\rho^2}$$

Let $f(z) = e^{i\delta}\left[\frac{z-p}{\overline{p}\,z-1}\right]$.

$$f'(z) = e^{i\delta}\ \frac{1\cdot(\overline{p}\,z-1)-(z-p)\,\overline{p}}{(\overline{p}\,z-1)^2} \text{ by the quotient rule}$$

$$= e^{i\delta}\ \frac{-1+|p|^2}{(\overline{p}\,z-1)^2}$$

$$|f'(z)| = |e^{i\delta}\ \frac{-1+|p|^2}{(\overline{p}\,z-1)^2}| = \frac{1-|p|^2}{|(\overline{p}\,z-1)^2|}, \text{ because } |e^{i\delta}| = 1, \text{ and } |p| \le 1.$$

$$|(\overline{p}\,z-1)^2| = |(\overline{p}\,z - 1)|^2 = (\overline{p}z-1)(p\overline{z}-1) = |p|^2|z|^2 - \overline{p}z - p\overline{z} + 1$$

At the boundary of the unit disc, $|z| = 1$, so

$$|(\overline{p}\,z-1)^2| = |p|^2 - \overline{p}z - p\overline{z} + 1$$

$$|z-p|^2 = (z-p)(\overline{z}-\overline{p}) = |z|^2 - \overline{p}z - p\overline{z} + |p|^2 = |p|^2 - \overline{p}z - p\overline{z} + 1$$

Hence $|(\overline{p}\,z-1)^2| = |z-p|^2$, so if $\rho = |z-p|$, $|f'(z)| = \frac{1-|p|^2}{\rho^2}$, which was to be shown.