Needham, Chapter 12, Exercise 18, p. 572.
G. Palmer, November 17, 2019, 1:20 pm

18 Use Harnack's inequality [previous exercise] to prove the following analogue of Liouville's Theorem: If T is harmonic in the whole plane and is bounded from above (or below), then T is a constant.

See the previous exercise for a figure related to Harnack's inequality. One can use the figure to determine where T(z) reaches minimums or maximums. Use Harnack's inequality to determine that minimums and maximums occur wherever r = R.

First assume that T(a) is positive everywhere on the complex plane or negative everywhere on the complex plane.

MIN MAX

$$0 = \left(\frac{R-r}{R+r}\right) T(0) \leq T(a) \leq \left(\frac{R+r}{R-r}\right) T(0) = \infty \text{ (or undefined)}$$

Positive T on $\mathbb{C}$ is bounded below by 0 and unbounded above.

Next assume that T(a) is negative everywhere on $\mathbb{C}$. For example, let $\tilde{T}$(a) = -T(a).

MIN MAX

$$-\infty = \left(\frac{R+r}{R-r}\right)\tilde{T}(0) \leq \tilde{T}(a) \leq \left(\frac{R-r}{R+r}\right)\tilde{T}(0) = 0$$

Negative T is bounded above by 0 and unbounded below. In both cases T(a) or $\tilde{T}$(a) is bounded by the point zero, which is interior to |z| = R. Now apply the Maximum-Modulus and Minimum-Modulus principles, explained on p. 357:

If f = u + iv is analytic then ... u and v are automatically "harmonic". Furthermore, *u and v are also subject to the principle that their* [?nonzero] *maxima and (nonzero) minima can only occur on* [the region boundary] *Γ, never inside Γ. As before, if a maximum or minimum occurs at an interior point, the harmonic function must be constant* [square brackets added].

Harnack's equation for T and $\tilde{T}$ shows both a maximum and a minimum at the interior point 0. We conclude that if T is harmonic on the whole complex plane, then T(a) is the constant 0. To reach this conclusion, the qualification "nonzero maxima" must be added to Needham's statement of the principle that maxima and minima can only occur on Γ.

A more general answer

Vasco has a more general solution in which T is not required to be either positive or negative on the entire plane. It is only required to have an upper or lower bounds. I have attempted a similar solution.

If T is bounded from above, then $T_{max} - T(\mathbf{a}) \geq 0$. Let $\mathcal{T}(\mathbf{a}) \equiv T_{max} - T(\mathbf{a})$. As the sum of two harmonic equations, $\mathcal{T}$ is a harmonic equation on the entire plane, so it must obey Harnack's equation.

$$\left(\frac{R-r}{R+r}\right)\mathcal{T}(0) \leq \mathcal{T}(\mathbf{a}) \leq \left(\frac{R+r}{R-r}\right)\mathcal{T}(0)$$

Since **a** has been chosen, we may now permit R to increase without limit. The result is that the quotients go to 1.

$$\mathcal{T}(0) \leq \mathcal{T}(\mathbf{a}) \leq \mathcal{T}(0)$$

This shows that $\mathcal{T}(\mathbf{a})$ is everywhere equal to $\mathcal{T}(0) = T_{max} - T(0)$, which is a constant by construction.

Similarly, if T is bounded from below, then $T(\mathbf{a}) - T_{min} \geq 0$. Let $\mathcal{T}(\mathbf{a}) \equiv T(\mathbf{a}) - T_{min}$.

$$\left(\frac{R-r}{R+r}\right)\mathcal{T}(0) \leq \mathcal{T}(\mathbf{a}) \leq \left(\frac{R+r}{R-r}\right)\mathcal{T}(0)$$

Since **a** has been chosen, we may now permit R to increase without limit. The result is that the quotients go to 1.

$$\mathcal{T}(0) \leq \mathcal{T}(\mathbf{a}) \leq \mathcal{T}(0)$$

This shows that $\mathcal{T}(\mathbf{a})$ is everywhere equal to $\mathcal{T}(0) = T(0) - T_{min}$, which is a constant by construction.

This shows that if T is bounded from above or from below on the complex plane, it must be a constant.

A revision to the question

Show that if T is harmonic in the whole plane, then T is a constant.

We begin with Harnack's equation

$$\left(\frac{R-r}{R+r}\right)T(0) \leq T(\mathbf{a}) \leq \left(\frac{R+r}{R-r}\right)T(0)$$

If **a** is interpreted to be any fixed point in the complex plane internal to a circle of any radius, then **r** is also fixed. Since T is harmonic on the whole complex plane, R may increase without limit, and no matter the value of **a** or **r**,

$$\left(\frac{R-r}{R+r}\right) \longrightarrow 1 \qquad \text{and} \qquad \left(\frac{R+r}{R-r}\right) \longrightarrow 1$$

so Harnack's equation reduces to

$$T(0) \leq T(a) \leq T(0) \tag{1}$$

Then it must be the case that T(a) = T(0) everywhere on the plane. And since T(0) is a constant by construction, T(a) must also be a constant, from which it follows that T is bounded both above and below.

This works only if **a** is chosen prior to chosing the radius R. If we decide R first, then no matter how large we make it, we can always find an **a** where |**a**| = **r** = R, from which we can draw no conclusion. Perhaps we could set R to be as large or small as we wish and then let r approach 0, which would recover (1). But since we are trying to show that T(a) is constant, our focus is on what happens at **a** (it becomes equal to T(0)), so we choose **a** first.