Needham, Chapter 12, Exercise 17, p. 572.
G. Palmer, November 11, 2019, 4:15 pm

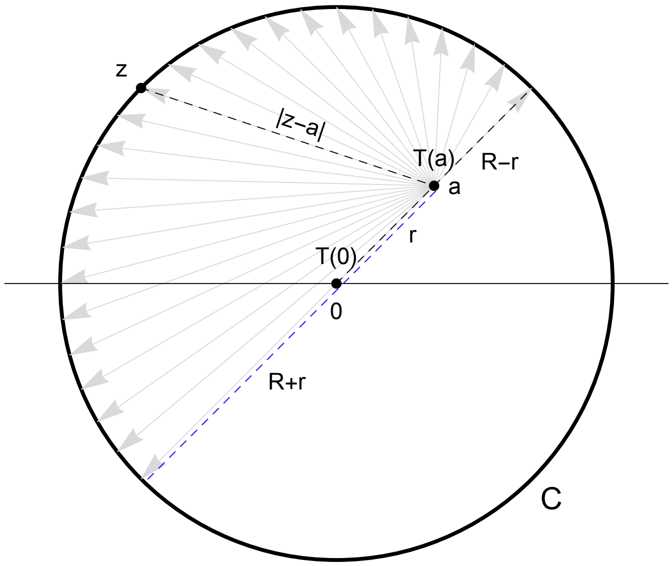

Figure 1. The disc

17 Let T(z) be a non-negative temperature distribution in the disk |z| ≤ R. Writing |a| = r, use Poisson's formula (28) to derive Harnack's inequality.

$$\left(\frac{R-r}{R+r}\right) T(0) \le T(a) \le \left(\frac{R+r}{R-r}\right) T(0)$$

From pp. 554-555 we learn that Poisson's formula (28) contains an "a" that can stand for "any interior point" and a "z" that moves around C of radius R. Given that $z = Re^{i\theta}$ and $a = re^{i\alpha}$, the formula is

$$\mathrm{T(a)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{R^2 - |a|^2}{|z-a|^2} \right] T(\theta)\, d\theta \qquad (28)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{R^2 - |a|^2}{R^2 + r^2 - 2\,\mathrm{Rr}\cos(\theta-\alpha)} \right] T(\theta)\, d\theta$$

The z in the problem statement can be any z with |z| ≤ R, but the z in (28) is constrained to C. In order to use (28) in the derivation, we confine z to the boundary. The θ of $z = Re^{i\theta}$ is the same as the θ in T(θ). The z mentioned in the problem statement is a more general z referring to all the complex points on the disc. We are interested in the T of a particular value of z on the disc, where z = a.

We imagine that z is tied to a by an elastic string (Figure 1). As z traverses C, the string contracts to its shortest length when $\theta = \alpha$, so that vector **z** is aligned with vector **a** and |z-a| = R-r. It stretches to its longest length when the vector **z** is again aligned with the the vector **a** at ($\theta = \alpha + \pi$) and |z-a| = R+r. The average Poisson weighting in (28) must occur somewhere between these two points. The point of maximum weighting occurs at minimum |z-a|. By the Poisson formula, this point has weighting

$$P_{\max} = \frac{R^2-|a|^2}{|z-a|^2} = \frac{R^2-r^2}{|z-a|^2} = \frac{(R-r)(R+r)}{(R-r)^2} = \frac{(R+r)}{(R-r)}$$

Similarly, the point of minimum weighting occurs at maximum |z-a|. This point has weighting

$$P_{\min} = \frac{R^2-r^2}{|z-a|^2} = \frac{(R-r)(R+r)}{(R+r)^2} = \frac{(R-r)}{(R+r)}$$

Then the weightings of all other points must fall between these two points. Where T(0) = $\langle T(\theta)\rangle$ for all points on C and r = |a|, we can write, only if T(z) ≥ 0 for all z on the disc,

$$\frac{(R-r)}{(R+r)}\mathrm{T}(0) = \frac{(R-r)}{(R+r)}\frac{1}{2\pi}\int_{-\pi}^{\pi}T(\theta)\,d\theta \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{R^2-|a|^2}{|z-a|^2}\right]T(\theta)\,d\theta$$
$$\leq \frac{(R+r)}{(R-r)}\frac{1}{2\pi}\int_{-\pi}^{\pi}T(\theta)\,d\theta = \frac{(R+r)}{(R-r)}\mathrm{T}(0)$$

By substitutions from equations (27) and (28), this reduces to

$$\left(\frac{R-r}{R+r}\right) T(0) \leq T(a) \leq \left(\frac{R+r}{R-r}\right) T(0)$$