Needham, Chapter 12, Exercise 16, p. 572.
G. Palmer, November 3, 2019, 5:16 pm

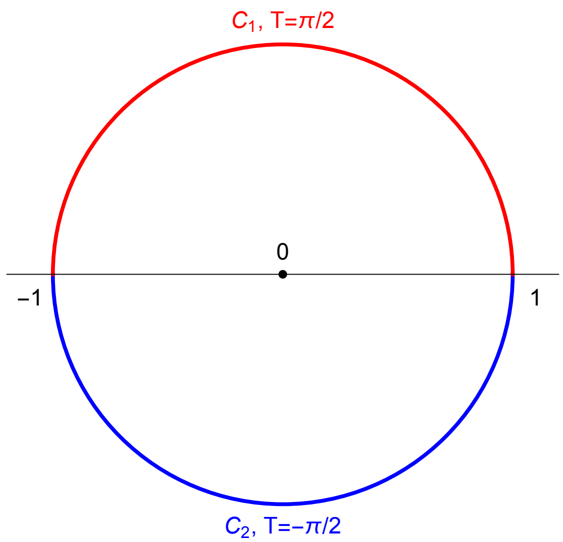

Figure 1a. The disc

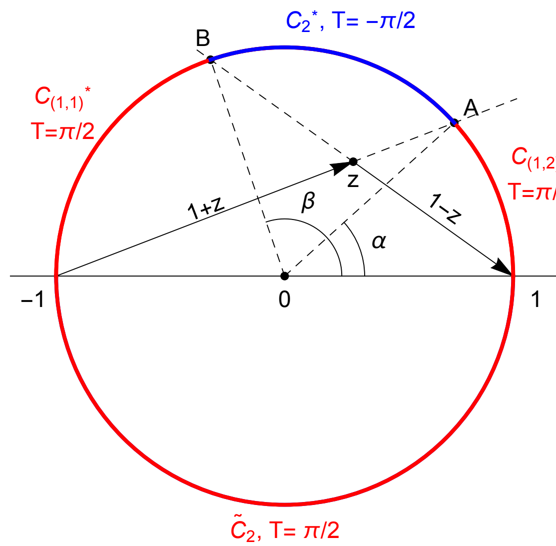

Figure 1b. The disc after Schwarz

16. Let T(z) be the temperature distribution in the unit disc if the top semicircle (Im(z) > 0) is kept at temperature +(π/2) and the bottom semicircle is kept at temperature -(π/2). Show that

$$T(z) = Arg\left[\frac{1+z}{1-z}\right]$$

If this problem is treated as a Dirichlet problem, then we can use Schwarz's interpretation:

To find the temperature at a, transplant each temperature on C to the point directly opposite to it as seen from a, then take the average of the new temperature distribution on C. (p. 556)

Then there will be a swap of C_2 into C_1. The new segment C_2^* completely replaces the temperature of that portion of C_1 with T = -π/2. At the same time, all the temperatures on C_2 (the bottom half circle) are replaced with T = π/2. In C_1 (the top half circle), the part not covered by C_2^* completes a swap of two segments ($C_{1,1}$ and $C_{1,2}$) that remain in C_1 with T = π/2. Since we are using a unit circle, the segments can be represented by their θ angles anchored at 0. For example, $C_1 = \pi = C_2$. The angle $C_2^* = (\beta\text{-}\alpha)$ can be found as a function of the quotient of the vectors (1-z) and (1+z). This function is not the same as the answer sought, but finding it is a key step.

Let α and β be the arguments of A and B on the unit circle C. Then C_2^* is the segment B-A on C with length equal to β-α. From the figure, we can write

$$\alpha = 2 \text{ Arg}(1+z)$$

$$\beta = \pi - 2\ (-\text{Arg}(1-z)) = \pi + 2 \text{ Arg}(1-z)$$

$$\beta - \alpha = (\pi + 2 \text{ Arg}(1-z)) - 2 \text{ Arg}[1+z] = \pi + 2 \text{ Arg}[(1-z)/(1+z)] = \pi - 2 \text{ Arg}[(1+z)/(1-z)]$$

We rewrite (30) using the parameters of the problem:

$$C_1 = \pi, \qquad C_2 = \pi, \qquad C_2^* = (\beta\text{-}\alpha) = \pi - 2 \text{ Arg}[(1+z)/(1-z)]$$

Integrate around the circle in the anticlockwise direction.

$$\text{T}(0^*) = \frac{1}{2\pi}\int_{C_{1,1}} \frac{\pi}{2}\, d\theta + \frac{1}{2\pi}\int_{C_2^*}\left(-\frac{\pi}{2}\right) d\theta + \frac{1}{2\pi}\int_{C_{1,2}} \frac{\pi}{2}\, d\theta\ \frac{1}{2\pi}\int_{C_2} \frac{\pi}{2}\, d\theta$$

$$= \frac{1}{4}\left(\int_0^{\alpha} d\theta - \int_{\alpha}^{\beta} d\theta + \int_{\beta}^{\pi} d\theta + \int_{-\pi}^{0} d\theta\right)$$

$$= \frac{1}{4}(\alpha - (\beta\text{-}\alpha) + (\pi - \beta) + \pi) = \frac{1}{4}\ (2\,\alpha - 2\,\beta) + 2\,\pi\big) = -\frac{1}{2}(\beta\text{-}\alpha) + \pi/2$$

$$= -\frac{1}{2}(\pi\text{-}2\text{Arg}[(1+z)/(1-z)]) + \pi/2 = \text{Arg}[(1+z)/(1-z)] - \pi/2 + \pi/2$$

$$= \text{Arg}[(1+z)/(1-z)]$$

T(0*) is the point T(z) that we are seeking. This is what we wanted.