Needham, Chapter 12, Exercise 15, p. 571.
G. Palmer, October 22, 2019, 4:30 pm

(1) Show that if u and v are harmonic then X = (u∇v - v∇u) is sourceless.

We assume that u and v are functions of z. We are to show that X is is sourceless (has a stream function). Then we must assume that X is a Pólya function.

Proof

We want to show that the flux density ∇•X equals 0, which is equivalent to saying the potential function is harmonic, i.e.,

$$\nabla \bullet X = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \bullet \begin{pmatrix} \partial_x \Phi \\ \partial_y \Phi \end{pmatrix} = \partial_x^2 \Phi \ + \ \partial_y^2 \Phi = \Delta\Phi = 0$$

(1) proof in the nabla notation. Apply the third product rule from the Wikipedia page on vector calculus identities:

$$\nabla \bullet X = \nabla \bullet (u\nabla v - v\nabla u)$$

$$= \nabla u \bullet \nabla v + u\nabla \bullet \nabla v - (\nabla v \bullet \nabla u + v\nabla \bullet \nabla u)$$

$$= (\nabla u \bullet \nabla v - \nabla v \bullet \nabla u) + (u\nabla \bullet \nabla v - v\nabla \bullet \nabla u) = 0$$

because the dot product is commutative: ∇u • ∇v = ∇v • ∇u; and because u and v are harmonic: ∇ • ∇ u **=** Δu = Δv = 0.

(2) proof in matrix notation

$$X = u \begin{pmatrix} \partial_x v \\ \partial_y v \end{pmatrix} - v \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} \quad \text{(pp. 497)}$$

$$= \begin{pmatrix} u\,\partial_x v - v\,\partial_x u \\ u\,\partial_y v - v\,\partial_y u \end{pmatrix}$$

$$\nabla \bullet X = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \bullet \begin{pmatrix} u\,\partial_x v - v\,\partial_x u \\ u\,\partial_y v - v\,\partial_y u \end{pmatrix}$$

$$= \partial_x u\,\partial_x v \ + \ u\,\partial_x^2 v \ - \ \left(\partial_x v\,\partial_x u \ + \ v\,\partial_x^2 u\right) \ +$$
$$\partial_y u\,\partial_y v \ + \ u\,\partial_y^2 v \ - \ \left(\partial_y v\,\partial_y u \ + \ v\,\partial_y^2 u\right)$$

$$= \ u\left(\partial_x^2 \ + \ \partial_y^2\right) v \ - \ v\left(\partial_x^2 + \partial_y^2\right) u = 0$$

because u and v are harmonic. Since the flux density ∇•X = 0, X must be sourceless.

(ii) Prove (37) by taking $u = T$ and $v = \mathcal{G}_a$, then equating the flux of X out of B with the flux out of an infinitesimal circle centred at a.

Prove: $\mathrm{T(a)} = \frac{1}{2\pi}\oint_B \mathcal{Q}_a(z)\, T(z)\,\mathrm{ds}$ (37)

Substituting for u and v in X = *(u∇v - v∇u) and letting* $\mathcal{Q}_a(z) = -\frac{\partial \mathcal{G}_a(z)}{\partial n}$ (p. 565), where $|\nabla \mathcal{G}_a(z)| = -\frac{\partial \mathcal{G}_a(z)}{\partial n}$, we obtain

$$\mathrm{X} = \mathrm{T(z)}\, \nabla \mathcal{G}_a(z) - \mathcal{G}_a \nabla T(\mathrm{z})$$

On p. 550 we have *We supply heat at a constant rate 2π to the point p of the region R, while holding the temperature all around the boundary B at the constant value 0. After the heat flow has settled down, the temperature in R will be a well defined (except at p) harmonic function* $\mathcal{G}_P(z)$ *called the Green's function of R with pole at p*. The unit disc provides an example, but on p. 553 we find that setting up a conformal mapping results in $\tilde{T}$[f(z)]=T(z), so , *In particular the values of T on B are transplanted to C*. It is true that no C is mentioned in exercise 15, but the construction of Green's function for the unit disc requires it (p. 551, paragraph ℙ. 1). $G_a(b) = 0$, where b is any point on B. Then the negative term containing $\mathcal{G}_a$ goes to zero, and we can write

$$\mathrm{X(b)} = T(\mathrm{b})\, \nabla \mathcal{G}_a(b)$$

This is a complex number. To find the flux out of B, we can write

$$\mathrm{F[X, B]} = \oint_B \mathrm{X(b)} \bullet \mathrm{N\, ds} = \oint_B [T(\mathrm{b})\, \nabla \mathcal{G}_a(b)] \bullet \mathrm{N\, ds}$$

Since Green’s function equals -Φ(z) (p. 551), and since $\mathcal{G}$ decreases away from the source, Φ must be positive, and the divergence $\nabla\mathcal{G}$ must have a positive value as z approaches the boundary. Then $\mathcal{Q}_a(b) = \left|\frac{\partial \mathcal{G}_a(b)}{\partial n}\right| = \left(-\frac{\partial \mathcal{G}_a(b)}{\partial n} \bullet N\right) = (\nabla \mathcal{G}_a(b)] \bullet \mathrm{N})$ is also a positive number. Letting b stand for any points on B, we can write

$$\mathrm{F[X, B]} = \oint_B T(\mathrm{b})\, \mathcal{Q}_a(b)\,\mathrm{ds} \qquad (1)$$

We can use Cauchy’s formula (p. 489) to calculate the flux out of a source inside an infinitesimal circle. We let T(z) = A, which is the real part of f(z). We omit the term -iB, which represents the vortex.

$$\oint_C \frac{f(z)}{(z-p)}\,\mathrm{dz} \equiv \oint_C \frac{T(a)}{(z-a)}\,\mathrm{dz} = 2\pi\mathrm{iA} = 2\pi\mathrm{iT(a)} \qquad (2)$$

Since this a source, which is all flux, we want just the imaginary part, the flux density for C (the circle, not the imaginary part of f(z)), which is 2πT(a). Then we set (1) equal to (2):

$$\oint_B T(b)\, \mathcal{Q}_a(b)\, \mathrm{ds} = 2\pi\ \mathrm{T(a)}$$

and the average flux density is

$$\mathrm{T(a)} = \frac{1}{2\pi} \oint_B \mathcal{Q}_a(z)\, T(z)\, \mathrm{ds}$$

which is the average of the boundary temperatures T(z). This is what we wanted.

[To be augmented with a second approach to calculating the flux out of the infinitesimal circle using the integral of X • N]

(iii) By taking u = $\mathcal{G}_a$ *and v =* $\mathcal{G}_b$*, prove the symmetry property of the Green's function:* $\mathcal{G}_a(b) = \mathcal{G}_b(a)$.

By comparison to part (ii), it appears that $\mathcal{G}_a$ and $\mathcal{G}_b$ are alternately taking the part of T(z).

$$X = \mathcal{G}_a \nabla \mathcal{G}_b - \mathcal{G}_b \nabla \mathcal{G}_a$$

Reasoning as in part (ii), and assuming that the strength of the flow is the same at points a and b,

$$2\pi\ \mathcal{G}_b(a) = \oint_B \mathcal{Q}_b(z) \mathcal{G}_a(z) \mathrm{ds}, \text{ because } \mathcal{G}_b = 0 \text{ on the boundary}$$

Making the opposite substitution into X, we have

$$2\pi\ \mathcal{G}_a(b) = \oint_B \mathcal{Q}_a(z) \mathcal{G}_b(z) \mathrm{ds}, \text{ because } \mathcal{G}_a = 0 \text{ on the boundary}$$

Then the total flux of X across B must be equal for flows emanating from a and b, so

$$\mathcal{G}_b(a) = \mathcal{G}_a(b)$$

This is what we wanted.