**Needham, Chapter 12, Exercise 14, p. 571**
**G. Palmer, September 16, 7:25 am PST**

*14 Show that applying the Milne-Thompson formula (17) to a vortex at a point p inside the unit circle yields two new vortices: at $1/\overline{p}$ and at 0. Explain this with the aid of a picture of the flow on the Riemann sphere.*

The key to this problem is recognizing that you can't get something from nothing, i.e. you can't get a new vortex by spontaneous regeneration. If a new vortex image shows up under a mapping, there must be a hidden preimage lying beyond the plot range of the computer software (and for quite some time, out of sight of my imagination). In this case, the origin-centred vortex in $\mathbb{C}$ has a hidden preimage in the flux $\Psi$ of the undisturbed complex potential. Streamlines of vortices at infinity are normally invisible on the complex plane ($\mathbb{C}$). But when a point in the neighborhood of infinity is stereographically projected to the Riemann sphere ($\Sigma$), it becomes visible in the neighborhood of the north pole (N). So it follows that Needham had in mind that we should find a way to project the hidden pre-image onto the sphere at N.

Even more generally, Vasco has pointed out (p.c. Forum) that a vortex centre, by which he means the streamlines centred on a point, does not occur in isolation. For a vortex centre in C, there is always a second vortex centre at infinity rotating in the opposite direction. Thus, if we have a vortex centre at the origin (call this $V_0$), there is a second ($V_\infty$) at infinity with opposite direction. By applying complex inversion*, $V_0$ maps to a conformal vortex ($V_{\infty \, \mathrm{Cf}}$) at infinity. $V_{\infty \, \mathrm{Cf}}$ can be projected to the neighborhood of N as $V_N$, where seen from the inside of the sphere, it is anti-conformal to $V_0$ and conformal to $V_\infty$. Hence, $V_N$ can be used to examine the behavior of $V_\infty$. Projections to the sphere are always conformal. Sources and sinks also have complementary vortices.

But there is a more direct way of seeing it without having to project it. Complex inversion* of the complex plane conformally maps the neighborhood of the origin to the neighborhood of infinity, and *vice versa*. Then all that is necessary to investigate the behavior of a vortex in the neighorhood of infinity is to apply complex inversion (f(z) = 1/z) to a neighborhood sufficiently distant from the neighborhood inside the unit circle. Then $\Omega(z)$ applied to this (1/z) neighborhood has streamlines that are conformal to those in the neighborhood of infinity, but they appear inside the unit circle.

A field vector for a vortex at a point p inside the unit circle can be written $\overline{H} = i/(\overline{z}-\overline{p})$, $|p| < 1$ (p. 488). One must first find $\Omega_u(z)$, discover its critical points, and then apply the M-T formula to see the new vertices. We know that the new vortex centre $1/\overline{p}$ is just p under inversion in C, which is built into the M-T formula. The other new centre at 0 must be $\infty$ under inversion in C. We would like to make the $\infty$-centred vortex visible near the north pole on $\Sigma$ by projecting the streamlines from the complex plane. Our plotting functions can not display objects near infinity, but given sufficiently large distances, sufficiently large radii of the circular streamlines in the complex plane, we can view projected streamlines near the north pole of the Riemann sphere. But the radii don't actually have to be very large. Vasco (p.c., Forum) has calculated how large they must be: *A circle at 0 with radius 10 in the complex plane has a radius of about 0.2 on the RS and a circle with radius 100 in the plane has a radius of 0.02 on the RS and so on. So all circles in the complex plane outside of the circle with radius 100, ie 100 to infinity lie inside the circle of radius .02 centred at the N pole. Another way of putting this is that all circles in the plane with radius greater than 3.75 lie* *within the circle of radius 0.5 centred on the N pole of the RS*. Perhaps these calculations also provide a way to used the second method described above to determine how far from the unit circle to map points back to the inside of the unit circle with complex inversion, thus avoiding projections altogether.

A field vector for a vortex at a point p inside the unit circle can be written $\overline{H}$ = i/($\overline{z}$-$\overline{p}$), |p| < 1 (p. 488).The Milne-Thompson formula is:

$\Omega_d(z) = \Omega_u(z)$ + Omega^†(z)   [Muliple subscripts and superscripts corrupt this file. Repair seems impossible.]

So we must first find $\Omega_u(z)$ and then apply the M-T formula.

$$H = \frac{-i}{z-p} = \text{Omega\_u(z)'}$$

$$\Omega_u(z) = \int \frac{-i}{z-p}\, dz = \text{-i Log(z-p)}$$

We apply the M-T formula f^†(z) = $\overline{f(1/\overline{z})}$ to find the disturbed potential.

$$\text{Omega\_u\^{}†(z)} = \overline{-i\,\text{Log}(1/\overline{z} - p)}$$

$$= \text{i Log}\left(\frac{1}{z}\,(1-\overline{p}z)\right)$$

$$= -i\text{Log}(z) + i\text{Log}(1-\overline{p}z)$$

$$= -i\text{Log}(z) + i\text{Log}\left(\overline{p}\left(\frac{1}{\overline{p}} - z\right)\right)$$

$$= \underset{\text{(at 0)}}{-i\text{Log}(z)} + \underset{\text{(at } 1/\overline{p})}{i\,\text{Log}\left(\frac{1}{\overline{p}} - z\right)} + \underset{\text{(const.)}}{i\text{Log}(\overline{p})}$$

The M-T result corresponds to a vortex at 0 and a vortex at $1/\overline{p}$ plus a constant. Both vortices appear to be new. Added to the undisturbed centre at p, we have centres p, $1/\overline{p}$, and 0. By complex inversion, the vortex centred at 0 must be accompanied by one additional complementary vortex opposite in direction and centred at infinity. (See, Ch 3, II, 1, p. 124, and III, 4, pp. 144-146). We can examine the properties of this vortex centre in either of two ways: (1) Apply complex inversion to the complex plane and examine the streamlines of -iLog(1/z) inside the unit circle; or (2) project streamlines of Omega_u^†(z) from outside the unit circle to the Riemann sphere. If the first method is followed, we know that complex inversion is conformal and it swaps the images in the neighbourhoods of the origin and infinity, so the image inside the unit circle C will have the same properties as those initially at infinity. The problem asks us to explain by using the Riemann sphere. Then we can use Vasco's calculations to project some of the original streamlines outside the unit circle onto the neighbourhood of N. The projected streamlines viewed from inside the sphere have the same direction as the preimage streamlines on the complex plane. They have the opposite direction to the original image at the origin viewed from above the plane.

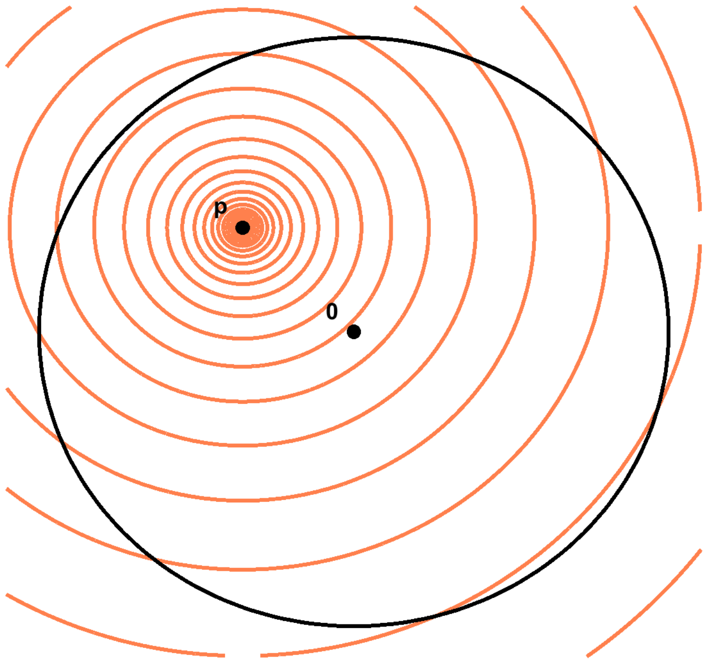


**Figure 1[a]. Streamlines of $\Omega_u(z)$**

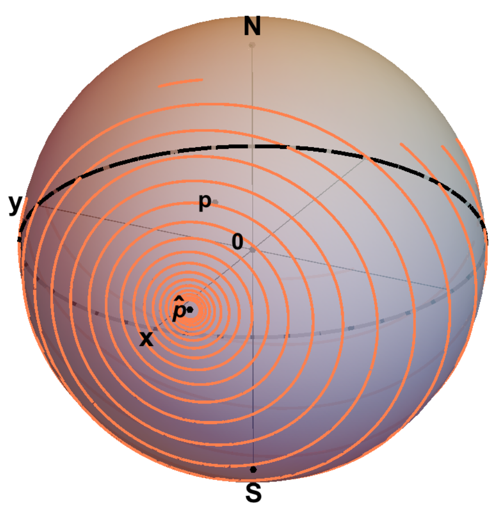


**Figure 1[b]. Streamlines of $\Omega_u(z)$**
**projected to $\Sigma$ in the southern hemisphere**

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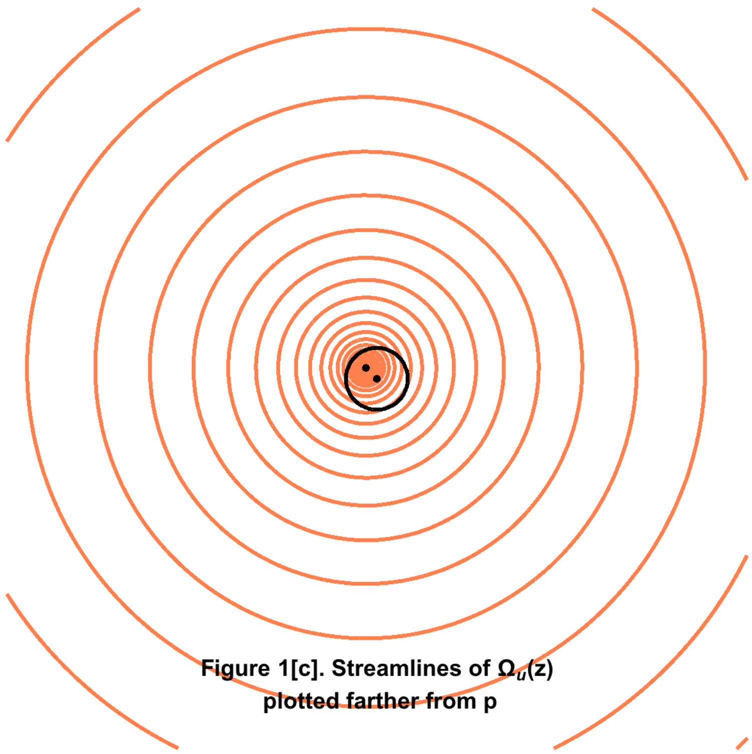

Figure 1[c]. Streamlines of $\Omega_u(z)$
plotted farther from p

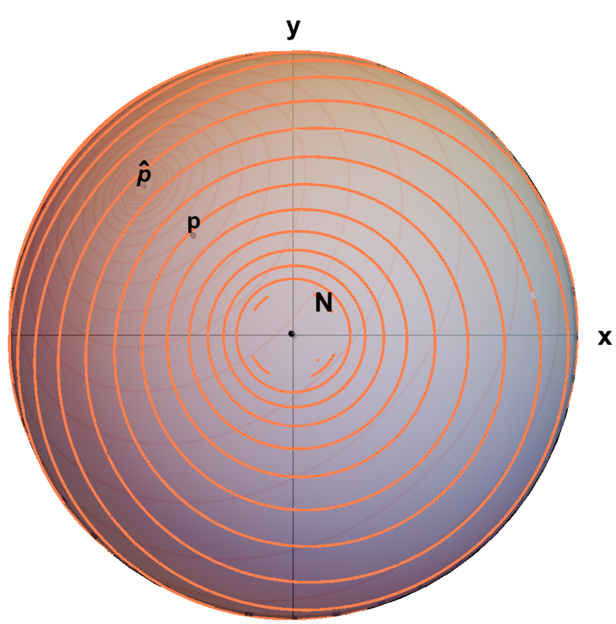


Figure 1[d]. Streamlines of $\Omega_u(z)$ projected to Σ
viewed from point over N

Figure 1[a] is the undisturbed flow of the vortex $\Omega_u(z) = -i \, \mathrm{Log}[z-p]$ centred at p = .6 Exp[I $3\pi/4$], plotted on ℂ and viewed from above. Streamlines inside the unit circle (black) project* to the southern hemisphere of $\Sigma$ in 1[b], and streamlines outside the unit circle project to the northern hemisphere. We know from the problem statement that there is a hidden vortex centred at infinity because it shows up in 2[a] as a vortex centred at 0, but it does not appear in 1[a] or 1[b] plotted with *Mathematica*'s ContourPlot. Following Vasco's insistence that a vortex centred at p inside C in the complex plane would require a complement centred at N on the sphere, the range of the x and y variables used by the contour plot were increased to the point where the content of the unit circle was no longer distinguishable. With these new parameters, one can see from 1[d] that there is indeed a vortex centred at $N = \infty$. The sphere is only about 80 percent opaque, so one can make out the very light streamlines round $p$, which is the projection of p onto the sphere in the southern hemisphere. I found this result completely unexpected, as I thought the vortex at p would have a greater effect on the vortex at N. Beyond a small skewing of the circles near the equator, there is not much visible difference from a single undisturbed vortex.

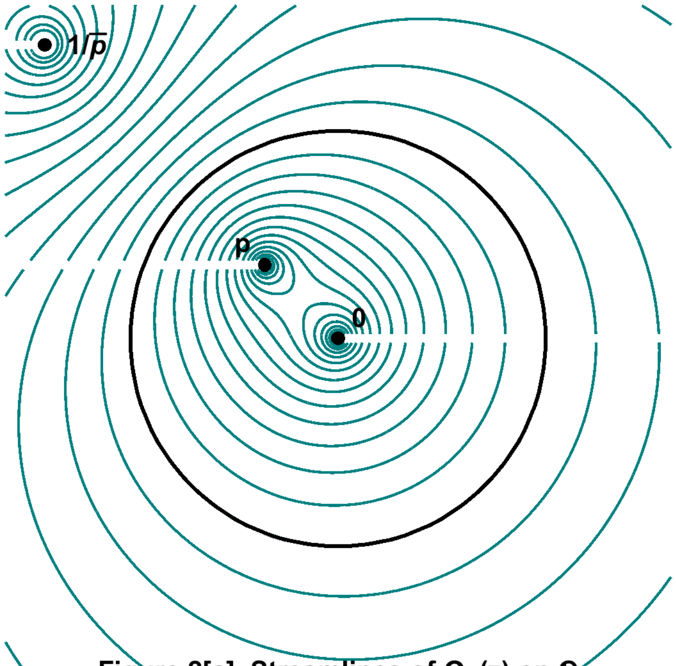


**Figure 2[a]. Streamlines of $\Omega_d$(z) on $\mathbb{C}$**

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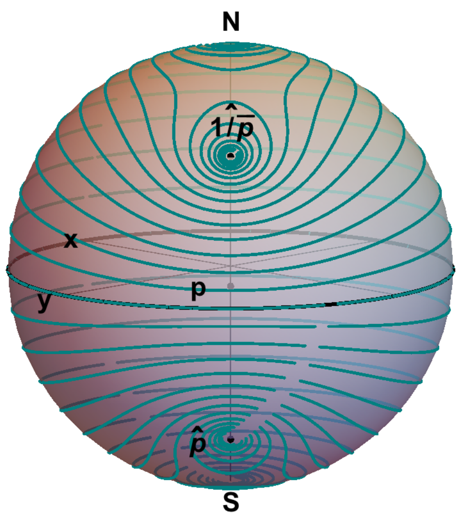


**Figure 2[b]. Streamlines of $\Omega_d$(z) projected to $\Sigma$**

Figure 2a shows the disturbed vortex plotted on $\mathbb{C}$ in the proximate neighbourhood of the origin. The vortex appears to be a composite of three vortices centred at 0, p, and $1/\overline{p}$. Figure 2[b] shows the disturbed vortex projected to $\Sigma$. The streamlines of 2[a] inside the unit sphere (black) are projected to the southern hemisphere in 2[b]. The projection $p$ of p appears in the southern hemisphere. The part outside the unit circle in 2[a] is projected to the northern hemisphere. The point of view in

2[b] is a point slightly above the complex plane and at the $\theta$-angle of p, which is on ℂ, as is $1/\overline{p}$, which is not shown. The projected point $1/\overline{p}$ is on Σ in the northern hemisphere. To answer the question, 2[a] shows the merging of three vortices centred at 0, p, and $1/\overline{p}$ on the complex plane. The projection to the sphere shows centres at p, $1/\overline{p}$, S, and N. The vortex at N = ∞ explains the new vortex at 0 in 2[a] by the principle of inversion in C. I find it most useful to think of two pairs of vortices centred at two inversion pairs in the complex plane: {p, $1/\overline{p}$} and {0, ∞}, which project onto the sphere at {p, $1/\overline{p}$} and {S=0, N=∞}. Hence, the figures explain the new vortices revealed by the algebra of the Milne-Thompson formula on the complex plane.

*"Projection" refers to stereographic projection. There are two sorts of inversion: (1) Complex inversion, which after projection onto Σ (the Riemann Sphere), rotates Σ by $\pi$ around the x axis (pp. 145-146). Complex inversion is conformal. For the definition, see p. 124. (2) Inversion in the unit circle on ℂ, which is incorporated into the Milne-Thompson formula. Inversion in C produces reflection about the equator on Σ. Inversion in C is anti-conformal. Inversion in C is the only type of inversion used in this answer.