Needham, Chapter 12, Exercise 13, p. 571
G. Palmer, August 13, 2019, 6:30 pm PST

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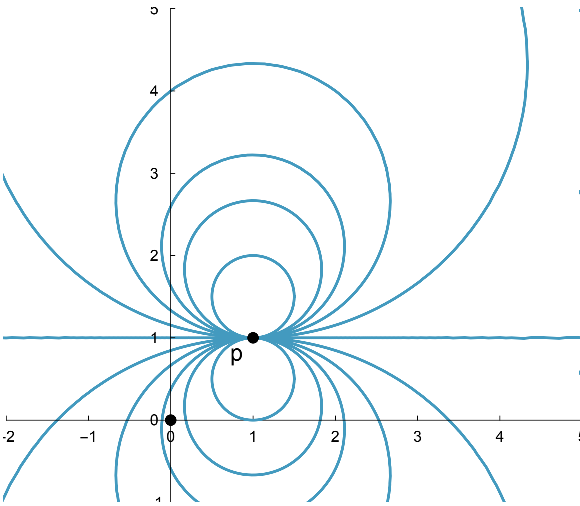

Figure 1. Dipole $\psi(z) = \mathrm{Im}[k/(p-z)]$

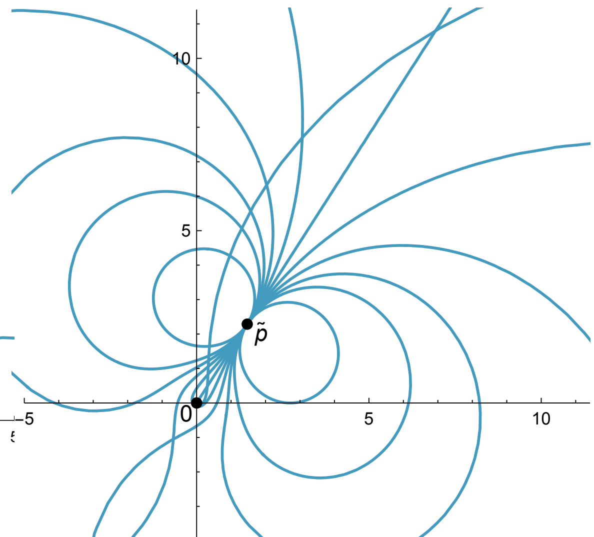

Figure 2. Dipole $e^z \circ \psi$

13 Repeat the investigation of the previous question in the case of a dipole located at p.

Consider the image under an analytic mapping f of a dipole of strength S located at p.

Rewrite the problem:

(i) Show geometrically, then algebraically, that if p is not a critical point of f (i.e. f'(p) ≠ 0) then the image is another dipole of strength (|f'| k) at f(p).

Geometric answer

The dipole (Figure 1) is the composition of a source and a sink formed from a doublet with centres separated at an infinitesimal distance on an axis (See Ch 12, Exercise 1). The source and sink are so close together that at our level of magnification we treat them as a single point p, but if they were actually equal, the doublet would self destruct and the field vector would vanish (p. 590). The dipole on a horizontal axis appears as nested circular streamlines of a source above p and a sink below. All the circles touch p. The streamlines of the source and sink flow in opposite directions. The dipole is symmetrical on both vertical and horizonal axes through p.

The strength of the dipole is somewhat arbitrary, because, in order to be observable, the strength S of the source must be increased in proportion to the diminishing separation as the poles of the doublet approach p (p. 491). The dipole in the image should have the same number of loops and k-flux tubes as the preimage, but they originate at f(p). Angles between circles are preserved near f(p). Infinitesimal circles near p go to infinitesimal circles near f(p). Given these considerations, near f(p), the image must be another dipole. See the algebraic answer for the explanation of why the flow in the tubes (or the electrostatic potential) must be increased as the separation between source and sink grows smaller. The result of applying an analytic function is to amplify (or contract) linear distances such as the distance ϵ between p and the source or sink. The strength k of the dipole is a simple multiple of this distance, so for a dipole under f(z), k must be a function of |f'|.

Algebraic answer

On p. 491, we find: *Putting $A = \epsilon e^{-i\phi} = -B$, the doublet field (1) becomes*

$$D(z) = \left[\frac{2\,\epsilon S e^{-i\phi}}{2\,\pi}\right]\frac{1}{\left(\bar{z}^2 - \epsilon^2\,e^{-i2\phi}\right)} \qquad (18)$$

which dies away as the source/sink separation 2ϵ tends to zero. The solution is to increase S in inverse proportion to 2ϵ, so that $2\epsilon S$ remains constant. He calls this real constant 2πk and writes the limiting doublet field (as $\epsilon \to 0$)

$$\mathrm{D(z)} = \mathrm{k}e^{-i\phi} \big/ \bar{z}^2$$

i.e. the general dipole field obtained by rotating [14] by $+\phi$ and scaling up the speed of flow by k, which we may think of as the "strength" of the dipole.

This raises questions: Why did Needham put scare quotes around "strength"? What is funny about this usage? Does k have the same units as S? Evidently not, because $2\pi k = 2\epsilon S$. Note also that $k = \epsilon S/\pi$ and $S = k\pi/\epsilon$. So k is S/π multiplied by a distance. Since S was defined as uniform charge per unit length, k must be an infinitesimal charge divided by π. Can we think of k as the charge at the point of coalescence? Vasco (Personal communication, Forum) regards k as flux per unit time by analogy with the definition of the strength of a source as units of fluid per unit time. The general dipole enclosed by a circle is both conservative and sourceless. Since $k = \epsilon S/\pi$, we can scale up the dipole by scaling up the S of the source and sink or by amplifying ϵ, but if we try to scale it up ϵ beyond an infinitesimal size, we are back to a doublet.

Then, on p. 548, Needham wrote about mapping sources and sinks into the unit circle. He said *Returning to [26]* [p. 544]*, the strength d of the dipole inside C is the amplification of f at p. The dipole of unit strength may be thought of as a doublet in which the source and sink are an infinitesimal distance ϵ apart, and each has strength $(1/\epsilon)$. The conformal mapping f sends this to another doublet at 0:* ***the strengths of the source and sink are preserved****, but their separation (and hence the strength of the dipole) is amplified by |f'(p)|....*

The strengths of the source and sink are preserved, but their separation and the strength of the dipole are amplified by |f'(p)|. That appears to be the solution. Note that on our previous calculations, we have treated the separation of the two centres of the source and sink as 2ϵ rather than ϵ as Needham has it in the foregoing quote. We continue with the original practice because it is more straightforward to write ϵ = (z-p). Symbols with tildes belong to the new dipole under f(z).

$$S = \frac{k\pi}{\epsilon} = \frac{k\,\pi}{\epsilon} = \frac{k\,\pi}{|f'|\,\epsilon} = \tilde{S}$$

$$\tilde{k} = \frac{k\pi}{\epsilon}\,\frac{|f'|\epsilon}{\pi} = |f'|\,k$$

The image is another dipole of strength (|f'| k) at f(p).

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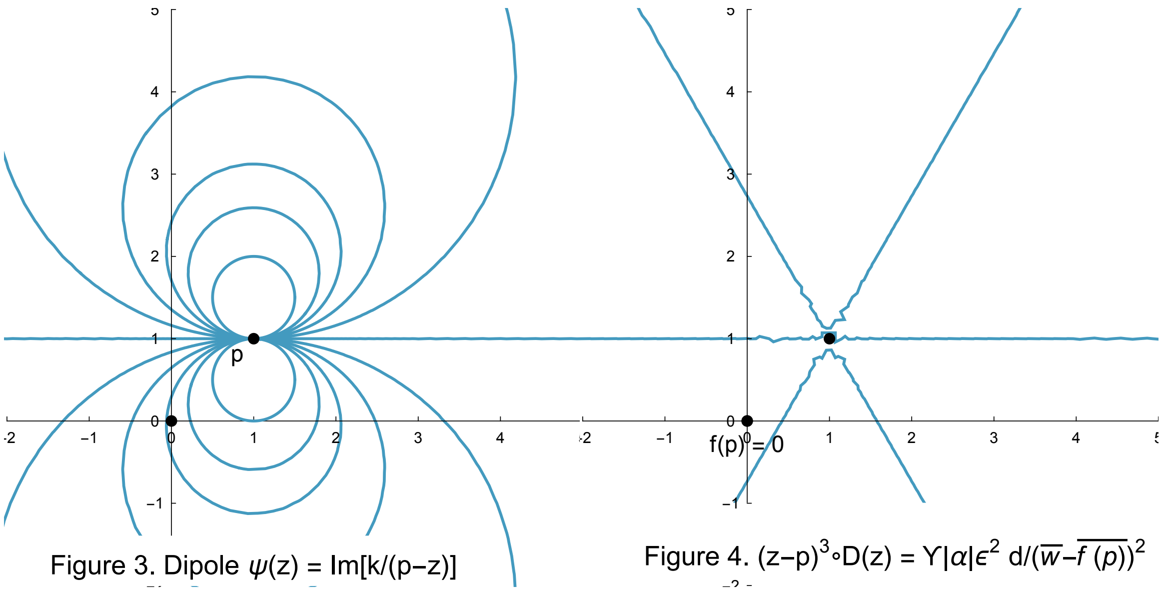

Figure 3. Dipole $\psi(z) = \mathrm{Im}[k/(p-z)]$

Figure 4. $(z-p)^3 \circ D(z) = \Upsilon|\alpha|\epsilon^2\, d/(\overline{w}-\overline{f(p)})^2$

Out[6178]=

Rewrite the problem:

(ii) *Show geometrically, then algebraically, that if p is a critical point of f of order (n-1) then the image is a new dipole of strength* $\tilde{k} = \frac{|\alpha|\,\Upsilon|\epsilon|^n S}{\pi}$ *and angle -nϕ.*

$$\tilde{D}(w) = \left[\,|\alpha|\;\Upsilon\,|\epsilon|^{n-1}\;|d|\,e^{-\mathrm{i}n\phi}\right] \big/ \left(\overline{w}-\overline{f(p)}\right)^2$$

Geometrical answer

Figure 3 is the same as Figure 1, showing the streamlines of a simple dipole. Figure 4 shows the streamlines in Figure 3 under the example function f(z) = $(z-p)^2$. Since f is analytic everywhere but at p, one might think that the conformality of f applied to the streamlines of Figure 3 should produce a dipole. But conformality breaks down at p as described on pages 205 and 249. Very near a critical point of f of order (n-1), angles of rays are multiplied by n (See p. 205, §§2, lines 6-7). At the critical point, amplification vanishes. [Figure 4 violates our expectations for angles multiplied by 3 at f(p). Since f(p) = 0, the rays should merge at 0, but they appear to merge at p. By inspection, they appear to be multiplied by twice what one would expect f to cause. Furthermore, every line has a flux of -π/1000.]

The source and sink are located at points we could label A and B. These are not critical points of f, so the strength S of source and sink remains unchanged under f. $\tilde{S}$ = S. The strength k of the dipole image under f depends on the amplification of the distance between the source and the sink under f.

Algebraic Answer

We modify the equation (18) on p. 491 to write ϵ as a vector, so we can write A = ϵ = $|\epsilon|\, e^{i\phi}$ and B = $-\epsilon$ = $-|\epsilon|\, e^{i\phi}$. The angle of the doublet in (18) is written as $e^{-i\phi}$ rather than $e^{i\phi}$ because D(z) is a Pólya vector field.

$$D(z) = \left[\tfrac{1}{2\pi}\left(2\,|\,\epsilon\,|\; S\, e^{-i\phi}\right)\right] 1 \Big/ \left(\overline{z}^2 - |\,\epsilon\,|^2\, e^{-2\,\phi}\right) \qquad \text{(18) modified}$$

Since our doublet is located at p = (A-B)/2, we can write

$$D(z) = \left[\tfrac{1}{2\pi}\left(2\,|\,\epsilon\,|\; S\, e^{-i\phi}\right)\right] 1 \Big/ \left((\overline{z} - \overline{p})^2 - |\,\epsilon\,|^2\, e^{-2\,\phi}\right)$$

Now we also have A = p + $|\epsilon|\, e^{i\phi}$ and B = p - $|\epsilon|\, e^{i\phi}$.

Then, letting the source/sink separation go to zero and letting the strength k of the dipole equal $\frac{|\epsilon|\,S}{\pi}$, we can write the dipole as

$$D(z) = k e^{-i\phi} \Big/ (\overline{z} - \overline{p})^2 = d \Big/ (\overline{z} - \overline{p})^2 = \left(|\,d\,|\, e^{-i\phi}\right) \Big/ (\overline{z} - \overline{p})^2$$

where ϕ is the direction of the dipole moment d and the axis.

We use the Taylor's series to find f(z) where p is a critical point of f and n ≤ 2 (p. 347), but substitute f(p) for p = f(a) and p for a.

$$f(z) - f(p) = f(p + \Delta) - f(p) = \frac{f'(p)}{1!}\Delta + \frac{f''(p)}{2!}\Delta^2 + \ldots,$$

Δ = (z-p) is, to paraphrase Needham, "a small complex number from p to a nearby point z". In the rare case where p is a critical point, as in this exercise, $f^{(n)}$ is the first nonvanishing derivative at p. Then from the equation, we can see that the dominant first term of the remainder of the series is proportional to Δ^n. Since we only need the dominant first term,

$$f(z) - f(p) = \frac{1}{n!} f^{(n)}(p)\, \Delta^n = \alpha\, (z - p)^n = \alpha \epsilon^n = \alpha \tilde{\epsilon} \qquad (1)$$

(1) is true in an infinitesimal neighborhood of p even when p is a critical point of f, which is the case in this exercise. Under f(z), a circle of radius ϵ is mapped to another circle of radius $|\,\alpha\,||\,\epsilon\,|^n$.

$$f(z) = \alpha\, (z - p)^n + f(p) = \alpha \epsilon^n + f(p)$$

$$\tilde{\epsilon} = (f(A) - f(p)) = (f(p+\epsilon) - f(p)) = \alpha \epsilon^n$$

$$-\tilde{\epsilon} = (f(B) - f(p)) = (f(p-\epsilon) - f(p)) = \alpha(-\epsilon)^n$$

$$\tilde{\epsilon} - (-\tilde{\epsilon}) = \alpha\epsilon^n - \alpha(-\epsilon)^n = \alpha\epsilon^n\,[1 - (-1)^n]$$

When n is even, the separation vector under f is 0. When n is odd, the separation vector is $2\alpha\tilde{\epsilon} = 2\,\alpha\epsilon^n$. Vasco wrote $(\tilde{A} - \tilde{B}) = \alpha\epsilon^n[1-(-1)^n]$. Letting $\Upsilon = [1-(-1)^n]$, $\Upsilon = 0$ when n is even and $\Upsilon = 2$ when n is odd. Since the following equations assume that 2 cancels in the numerator and denominator, I will let $\Upsilon = 1$ when n is odd. Then

$$d = |d|\, e^{-i\phi}$$

$$|\tilde{d}| = \tilde{k} = \frac{|\epsilon| S}{\pi} = \frac{1}{\pi}\left|\alpha\right| \Upsilon \left|\epsilon\right|^n S \qquad (\tilde{S} = \text{S, because source and sink are non-critical points})$$

$$\tilde{d} = |\tilde{d}| e^{-in\phi}$$

Then

$$\tilde{D}(w) = \tilde{d} \Big/ \left(\overline{f(z)} - \overline{f(p)}\right)^2 = \left[\frac{1}{\pi}\left|\alpha\right| \Upsilon \left|\epsilon\right|^n S\, e^{-in\phi}\right] \Big/ \left(\overline{w} - \overline{f(p)}\right)^2 = \left[\Upsilon\left|\alpha\right|\left|\epsilon\right|^{n-1}\ \left|d\right| e^{-in\phi}\right] \Big/ \left(\overline{w} - \overline{f(p)}\right)^2$$

$$= \left[\Upsilon\left|\alpha\right|\epsilon^{n-1}\, d\right] \Big/ \left(\overline{w} - \overline{f(p)}\right)^2$$

This is a new dipole of strength $\tilde{k} = \frac{1}{\pi}(\Upsilon\left|\alpha\right|\ \left|\epsilon\right|^n S)$ and angle -nϕ. The effect of Υ is to make the dipole vanish whenever n is even! Is there any parallel to this in nature? The dipole under this f(z) vanishes more quickly than the preimage dipole by the power n. This means that the dipole can only be maintained by multiplying S by a power of n. One might say that this dipole is very timid. If S is constant, it will quickly vanish with decreasing ϵ, even more quickly with larger n. Then only a very large increase in S can coax the dipole back into the light. One might say that the dipole has a vanishing coefficient Υ and a vanishing power n. Like the general dipole, this dipole vanishes completely at $\epsilon = 0$.