Needham, Chapter 12, Exercise 12, p. 571
G. Palmer, October 8, 2019, 6 pm PST

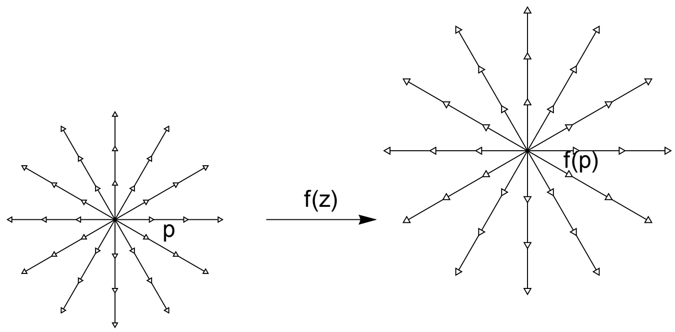


Figure 1. Schematic of source

12 Consider the image under an analytic mapping f of a source of strength S located at p.

(i) Show geometrically, then algebraically, that if p is not a critical point of f (i.e. $f'(p) \neq 0$) *then the image is another source of strength S at f(p).*

The Geometric Explanation

Figure 1 displays the transformation first imagined on reading part (i), based on a source such as that shown in [12a], p. 489, transformed to another source of the same strength. We take "strength S" to mean a scalar component of flow emanating from a source point. It is not the same as flux across a simple loop. But in the case that a source point is encircled by a simple loop, then the flux across the loop equals S given a certain conventional description of the source vector field, i.e. $(1/2\pi)\ (1/(\bar{z}\text{-}\bar{p}))$. On p. 547, we find *Since the strength of a source is measured by the number of k-flux tubes emanating from it, a conformal mapping sends a source to another source of equal strength*. To show this geometrically, we can plot the source in the z-plane and then apply an analytic function to the points of the preimage. The source in the image should have the same number of evenly spaced rays and k-flux tubes as the preimage, but they originate at f(p). In other words, every point z goes to a new point w in such a way that continuity and angles are preserved up to the point f(p). Figure 1 shows 12 k-flux tubes and 12 streamlines (rays) for a source of strength S, so each tube has flux k = S/12. Since the image has the same number of flux tubes separated by the same constant angle, the flux must be equal. That is the transformation as I imagined it, and I think it is an adequate approximation where z is very close to p. But the image is misleading as a depiction of the general case, because the vast majority of analytic functions would bend the rays. See Figure 2, for example in which the rays in [a] are transformed under analytic f(z) = e^z into the curved lines in [b]. Figure 2 illustrates the conformal property of the analytic function near the source; all the intersections of streamlines with the dashed lines of potential are orthogonal. Farther from the source than is shown in [b], there are non-orthogonal intersections. This conformality near the source means that the angles and number of flux tubes and the angles between the streamlines are equal for source and image. Hence, we can still say that the flux is equal for preimage and image. Vasco has pointed out (Forum, personal communication) that the amplitwist just rotates the 'cartwheel' of streamlines and flux tubes about p and makes the 'spokes' longer or shorter, so the fluid flow along the tubes is unaltered. Hence, the strength of the flow is also unaltered.

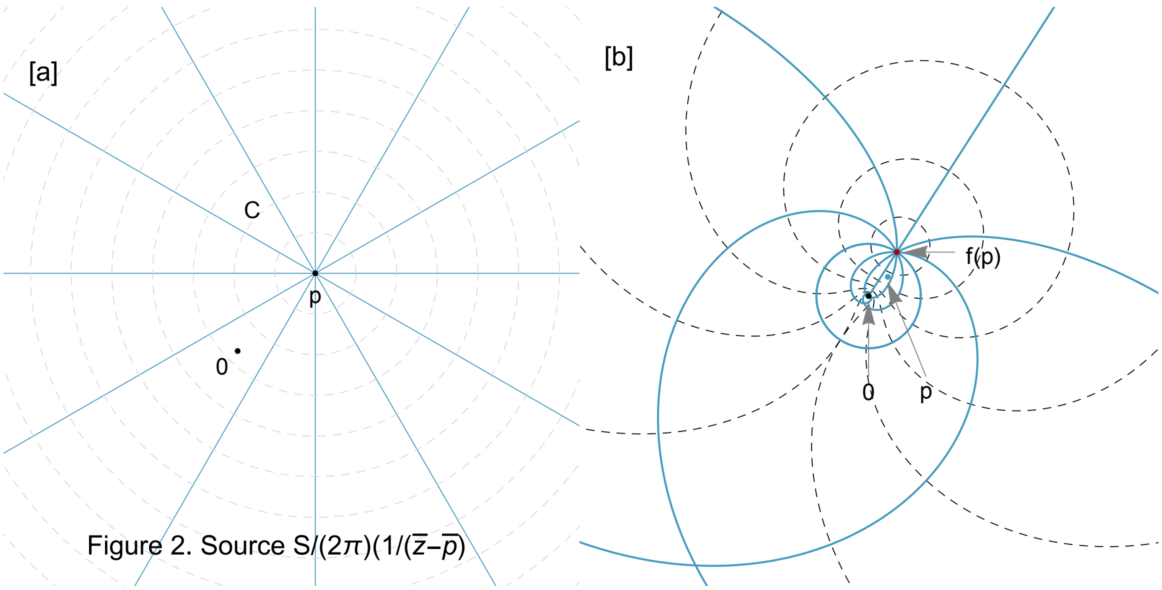
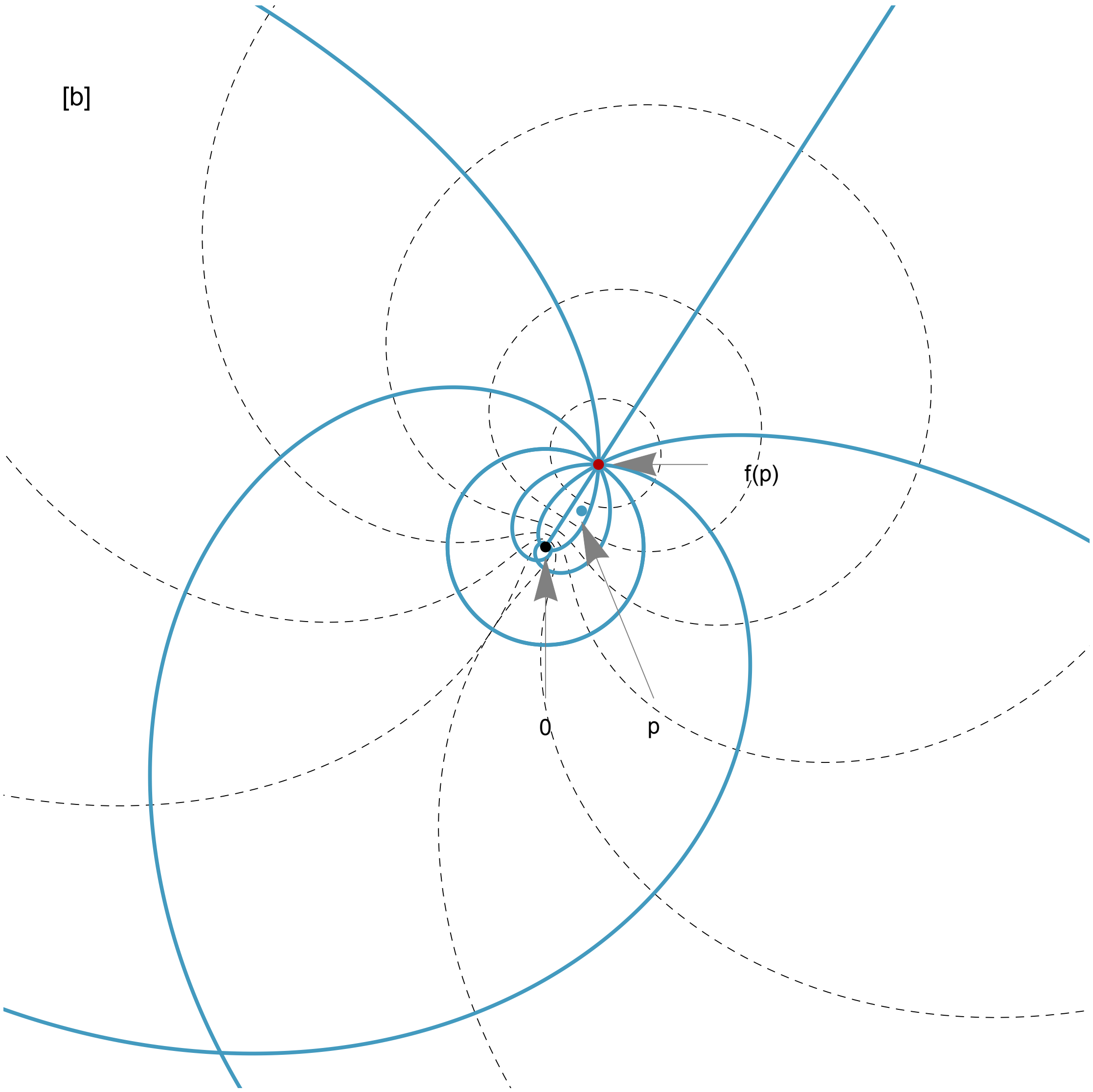


Figure 2. Source $S/(2\pi)(1/(\overline{z}-\overline{p})$



An algebraic explanation using F[$\overline{H}$, C] = $\int_K$X•N ds

On p. 485, we find "the Polya vector field $(1/\overline{z})$ is an old friend—it is a source of strength 2π located at the origin." This suggests that we can write $\overline{H}(z) = (S/2\pi)(1/(\overline{z}-\overline{p})$ to describe the source of strength S centred at p. To paraphrase Needham, "if a simple loop does enclose the source, it intercepts the full S of fluid being pumped in at the origin". Normally the word "intercept" means to stop or cut off a movement, which might lead one to conclude that the flux across a closed loop is zero, in which case a loop integral would be pointless. But Needham seems to be using the term "intercept" to mean "intersect" without stifling the flow. Perhaps it is a typographical error. Vasco asserts that there is flux across a closed loop surrounding a source, and Needham states that a source is "all flux", which entails that it crosses a loop. *Ergo*, a source induces flux across a loop.

The problem implies that in some circumstances, strength $\tilde{S}$ of the image vector field differs from S of the preimage vector field, but in this case they are equal. The problem is to prove it. We can do this by finding and solving an equation that contains both S and $\tilde{S}$. Since p is not a critical point of f, an infinitesimal single loop under f goes to another infinitesimal single loop. The geometric proof shows that the fluxes of the image and preimage over their respective infinitesimal loops are equal, and we use that fact to establish algebraically that the strengths are equal. To calculate flux, I follow Vasco's lead and use $F[X, K] = \int_K X \bullet N \, ds$ from p. 473. The radial streamlines very close to a source at p are parallel to the normal (a line segment orthogonal to an infinitesimal circle centred at p) with the nice result that $X \bullet N = |X|$. I retain the use of $\overline{H}$ for the vector field in place of X just to keep it embedded in a field of connotations regarding Pólya vector fields and sources. Our equation is

$$F[\overline{H}, C] = F[\overline{\tilde{H}}, \tilde{C}] \tag{1}$$

where

$$\overline{H}(z) = \frac{S}{2\pi} \frac{1}{\bar{z} - \bar{p}},$$

C is a circle of infinitesimal radius $|\epsilon| = |z-p|$ centred at p, and $\tilde{C}$ is a circle of infinitesimal radius $|\tilde{\epsilon}| = |f'(p)\, \epsilon| = |f(z) - f(p)|$ centred at f(p). (*To obtain the image at f(z) of an infinitesimal complex number at z, you just multiply it by f'(z))*. P. 198). Since this is a loop integral, it is convenient to rewrite ds in terms of $d\theta$. Equation (1) will be justified below.

$$\overline{H}(z) = \frac{S}{2\pi} \frac{1}{\bar{\epsilon}}, \qquad ds = |\epsilon| \, d\theta$$

$$F[\overline{H}, C] = \int_C \overline{H} \bullet N \, ds = \int_C \left| \overline{H}(z) \right| ds = \int_0^{2\pi} \frac{S}{2\pi} \frac{1}{|\epsilon|} \left| \epsilon \right| d\theta = S$$

This shows that on a single loop, the flux of a source equals the strength "S" of its conventional vector field $\frac{S}{2\pi} \frac{1}{\bar{z} - \bar{p}}$.

The streamlines are rays as seen in Figure 2[a]. Apply f(z) to the points of the streamlines to obtain the streamlines of the new source. Very close to f(p) the analytic function sends C to $\tilde{C}$. Tildes (e.g. $\tilde{H}, \tilde{\epsilon}, \tilde{C}, d\tilde{s}$) indicate the designated object belongs to the plane of f(z).

$$d\tilde{s} = |\tilde{\epsilon}| \, d\tilde{\theta}$$

$$\overline{\tilde{H}}(z) = \frac{\tilde{S}}{2\pi} \frac{1}{\bar{\tilde{\epsilon}}}$$

$$\overline{\tilde{H}} \bullet N = |\overline{\tilde{H}}| = \frac{\tilde{S}}{2\pi} \frac{1}{|\tilde{\epsilon}|}$$

$$F[\overline{\tilde{H}}, \tilde{C}] = \frac{\tilde{S}}{2\pi} \int_{\tilde{C}} \frac{1}{|\tilde{\epsilon}|} \left| \tilde{\epsilon} \right| d\tilde{\theta} = \frac{\tilde{S}}{2\pi} \int_0^{2\pi} d\tilde{\theta} = \tilde{S}$$

Since we know that the fluxes are equal, we see that $\tilde{S} = S$. This verifies that the strength of the image source at f(p) is S.

(ii) *Show geometrically, then algebraically, that if p is a critical point of f of order (n-1) then the image is again a source, but now of strength (S/n).*

Geometrical answer

If p is a critical point of f of order (n-1), then if n ≥ 2, f becomes nonconformal at p. To paraphrase Needham (p. 205), rather than being conserved, angles at the critical point ... are consequently muliplied by n. This means that every flux tube expands its diameter by a factor of n. Hence, the image of the source under f(z) has fewer flux tubes by a factor of (1/n) per traversal of f(z) round the image circle $\tilde{C}$. The radius of a an infinitesimal circle under f(z) and centred at p shrinks by a power of n to r = ϵ^n. The circumference shrinks to $2\pi\epsilon^n$. But, the shrinkage of radius and circumference have no effect on flux. If the image be said to have strength S/n, it must be because the number of flux tubes has been divided by n, assuming that an expanded flux tube in the image carries the same amount of flux as the flux tube in the preimage.

But the story may be more complicated than this, because the loop of f(p) round $\tilde{C}$ should make n winds. Since the flux tubes are expanded, after the first wind, tubes in the neighborhood of f(p) must be superimposed on tubes from earlier winds. Each loop has fewer flux tubes, but the flow in each flux tube remains the same. Since there are fewer flux tubes per loop but more loops, the number of flux tubes remains the same. We can deduce that the total flux of the image adding flux from the larger tubes in all the loops equals the total flux of the preimage. Then if the flux of a single, visible loop in the image is $\tilde{S}$, we can conclude that S = n$\tilde{S}$ and $\tilde{S}$ = S/n, as was to be shown. Or as Vasco puts it, as the infinitesimal diameter of the preimage winds around p once, the amplified infinitesimal diameter of the image winds n times around f(p), capturing 1/n of the flux with each wind.

Algebraic answer

The most simple f(z) that meets the conditions is f(z) = $(z-p)^n$, n ≥ 2. Taylor's series can be used to calculate f(z) near p.

$$f(z) = f(p) + \frac{f^{(n)}(p)}{n!}(z-p)^n$$

$$= (p-p)^n + \frac{n!}{n!}(z-p)^n \quad = 0 + \frac{n!}{n!}(z-p)^n = \epsilon^n$$

$$\tilde{\epsilon} = f(z) - f(p) = f(z) = \epsilon^n$$

$$d\tilde{s} = |\tilde{\epsilon}|\, d\,\tilde{\theta}$$

$$\left|\ \overline{\tilde{H}}\ \right| = \frac{\tilde{S}}{2\pi}\frac{1}{|\tilde{\epsilon}|}$$

$$F[\overline{\tilde{H}}, \tilde{C}] = \int_{\tilde{C}} |\overline{\tilde{H}}|\, d\,\tilde{s} = \frac{\tilde{S}}{2\pi}\int_0^{\mathbf{n2\pi}} \frac{1}{|\tilde{\epsilon}|}\left|\tilde{\epsilon}\right| \boldsymbol{d\,\tilde{\theta}} = n\tilde{S}$$

With d$\tilde{\theta}$ we explicitly integrate over n winds. From part (i), we know that the preimage source has strength S. The total flux of an image loop under n winds is n$\tilde{S}$. It seems in order to solve the problem we must again write (1).

$$(\mathrm{F}[\overline{H}, \mathrm{C}] = \mathrm{F}[\overline{\tilde{H}}, \tilde{C}]) \Longrightarrow (\mathrm{S} = \mathrm{n}\tilde{S}) \Longrightarrow \tilde{S} = \mathrm{S/n}, \text{ which is what we wanted.}$$

This is my version after having seen Vasco's answer. Vasco uses a more sophisticated application of Taylor's series.

An algebraic excursion using Cauchy's formula (§§6, pp. 488-489)

The problem asked us to transform the image of the source, which involves applying f(z) to the points of the preimage, that is, the points of the rays that represent the streamlines and the concentric circles that represent the potentials. What would happen if we instead multiply the vector field by f(z)? Using Cauchy's formula, we can discover the effect on S.

Drawing on Needham's discussion of Cauchy's formula (p.), "If f(p) = A + iB, then very close to p the Polya vector field $\overline{H}$ will be indistinguishable from

$$\frac{A-\mathrm{iB}}{\overline{z}-\overline{p}} \ldots\text{"},$$

which composes a source and a vortex. For our purposes, we can say that if f(p) = A where A is a real analytic function, then very close to p the Pólya vector field $\overline{H}$ will be indistinguishable from A(S/2π)(1/($\overline{z}$-$\overline{p}$)), which is a source of strength AS at p. (Because if 1/(z-p) is known to have strength 2π, then a source of strength S should be written (S/2π)(1/($\overline{z}$-$\overline{p}$)). Needham multiplies the original strength by A: *the first term is familiar as a source at p of strength 2πA (p. 488, line -4).** The first term to which he refers is A[1/($\overline{z}$-$\overline{p}$)]. We can write our first term as

$$\mathrm{A}\left[\frac{S/(2\,\pi)}{\overline{z}-\overline{p}}\right]$$

Using Cauchy's formula from p. 430, we can integrate this term to obtain iF[$\overline{H}$, C].

$$\oint_L \frac{f(z)}{z-a}\mathrm{dz} = 2\pi\mathrm{i}\nu[\mathrm{L, a}]\ \mathrm{f(a)}\ \text{[from p. 430]}$$

Adapting this to our problem, we write

$$(\mathrm{S}/2\pi)\oint_L \frac{f(z)}{z-p}\mathrm{dz} = (\mathrm{S}/2\pi)\ 2\pi\mathrm{i}\nu[\mathrm{C, p}]\ \mathrm{f(p)} = \mathrm{i\ S\ f(p)} = \mathrm{iF}[\overline{H}, \mathrm{C}]$$

We can write

$$\frac{S}{2\,\pi}\oint_C \frac{A}{(z-p)}\mathrm{dz}$$

$= \frac{S}{2\pi}\, i\, 2\pi\, A = iS \cdot A$

This apparently shows that the amplified vector field describes a source (? at f(p)) with flux (strength) S·f(p), but only if f(p) is a real analytic function. If f(p) is a complex analytic function, a vortex is also created.

*In the case of f(p) = e^z, we can write $e^z = e^x e^{iy} = e^x \cos(y) + i\, e^x \sin(y)$. Then A = $e^x \cos(y)$ and the flux (strength) under f(z) is (S e^xcos(y)). At p = 1+i, as in Figure 2, this becomes $F[\overline{\tilde{H}}, \tilde{C}] =$ $Se^1 \cos(1) \approx S\,(.54)\, e = 1.47\, S$.