Needham, Chapter 12, Exercise 11, p. 571
G. Palmer, February 25, 2019, 10 am PST

11 With the same notation as in the previous question, let p measure distance in the direction $i\hat{\xi}$ perpendicular to the curve. By substituting $\mathcal{K} = -i\nabla\Psi$ into (30), p. 239, show that the image curvature is given by the tidy formula,

$$\tilde{\kappa} = \partial_p \Psi + \kappa \Psi.$$

By the same notation, $\Psi = 1/|f'|$. Now p in the direction $i\hat{\xi}$ perpendicular to the curve is measured along the radius of curvature.

$$\tilde{\kappa} = \mathcal{K} \bullet \hat{\xi} + \frac{\kappa}{|f'|} \qquad (30)$$

Rewrite (30) at a point α.*

$$\tilde{\kappa}(\mathrm{p}) = (-i\,\nabla\,\Psi\,(\alpha)) \bullet \hat{\xi}(\alpha) + \kappa(\alpha)\Psi(\alpha) \quad \text{by substitutions}$$

$$= \nabla\,\Psi(\alpha) \bullet i(\hat{\xi}\,(\alpha)) + \kappa(\alpha)\Psi(\alpha) \quad \text{by Hint, Ex. 10 (iii)}$$

The term $\nabla\,\Psi(\alpha) \bullet i\,\hat{\xi}\,(\alpha)$ represents the vector $\nabla\Psi$ in the direction of maximum change of Ψ projected onto the radius of curvature in the direction of $i\hat{\xi}$. Since p has the direction $i\hat{\xi}$ by construction, this is the change of Ψ on p, or $\partial_p\Psi$. Hence,

$$\tilde{\kappa} = \partial_p \Psi + \kappa\Psi$$

Vasco's answer demonstrates how to calculate $\partial_p\Psi$, so I present my understanding of it. In the discussion of the gradient field on p. 497, we find

$$d\Psi = \boldsymbol{\nabla\Psi} \bullet \mathbf{dz}$$

The vector **dz** can here be rewritten $i\hat{\boldsymbol{\xi}}$dp.

$$d\Psi = \boldsymbol{\nabla\Psi} \bullet i\hat{\boldsymbol{\xi}}dp$$

$$\frac{d\Psi}{dp} = \boldsymbol{\nabla\Psi} \bullet i\hat{\boldsymbol{\xi}} \qquad \text{dividing through by p}$$

Then, $\frac{d\Psi}{dp} = \frac{\partial\Psi}{\partial p} = \partial_p\Psi = \boldsymbol{\nabla\Psi} \bullet i\hat{\boldsymbol{\xi}}$

We can write $\frac{d\Psi}{dp} = \frac{\partial\Psi}{\partial p}$ because the ratio of actual changes obeys the rule specified by the ratio of infinitesimals $\frac{\partial \Psi}{\partial p}$.

*We are accustomed to thinking of “p” as a point on a curve. Here it is evidently a distance **from** a point (“p” by our previous notation) on a curve. The distance is measured along the radius of curvature. This is a bit confusing. We might instead write the equivalent $\partial_r \Psi$, where r is the radius of curvature, or $\partial_\rho \Psi$ to distinguish the radius of curvature from |z|.