**Needham, Chapter 12, Exercise 10, p. 571.**
**G. Palmer, February 24, 2019, 1:30 pm PST**

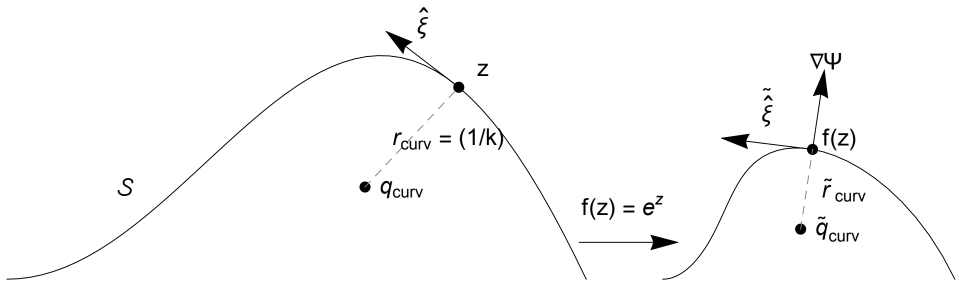


Figure 1. Some features of Exercise 10, Chapter 12.

*10 As usual, let $\hat{\xi}$ be the unit tangent to a curve of curvature $\kappa$, and let $\tilde{\kappa}$ be the curvature of the image under the analytic mapping f(z). Also let s and $\tilde{s}$ denote arc length of the original and image curves. Finally, let $\Psi = (1/|f'|)$ be the stream function of the complex curvature, $\mathcal{K} = -i\nabla\Psi$.*

*(i) Use (30), p. 239 to show that* $\partial_{\tilde{s}}\tilde{\kappa} = \Psi\,\partial_s\left[\kappa\Psi + \mathcal{K}\bullet\hat{\xi}\right]$.

By (30), p. 239, $\tilde{\kappa} = \mathcal{K}\bullet\hat{\xi} + \frac{\kappa}{|f'|}$. Then by substitution, $\tilde{\kappa} = \mathcal{K}\bullet\hat{\xi} + \kappa\Psi$.

From p. 198, we find that the effect of an analytic function on an infinitesimal disc (or vector) centred (or anchored) at z is to amplify and twist it, an effect which is "completely encoded in the single complex number f'(z) = |f'(z)| $e^{i\arg[f'(z)]}$. On p 237, (26), we find $|\tilde{\xi}| = |f'(p)| \bullet |\xi|$, showing that a scalar in the z plane under f is amplified by |f'(p)|. Then we can write:

$$\partial_s\tilde{\kappa} = |f'|\,\partial_{\tilde{s}}\tilde{\kappa} \text{ and } \partial_{\tilde{s}}\tilde{\kappa} = \frac{1}{|f'|}\partial_s\tilde{\kappa} = \Psi\,\partial_s\tilde{\kappa} = \Psi\,\partial_s[\kappa\Psi + \mathcal{K}\bullet\hat{\xi}] \quad \text{QED}$$

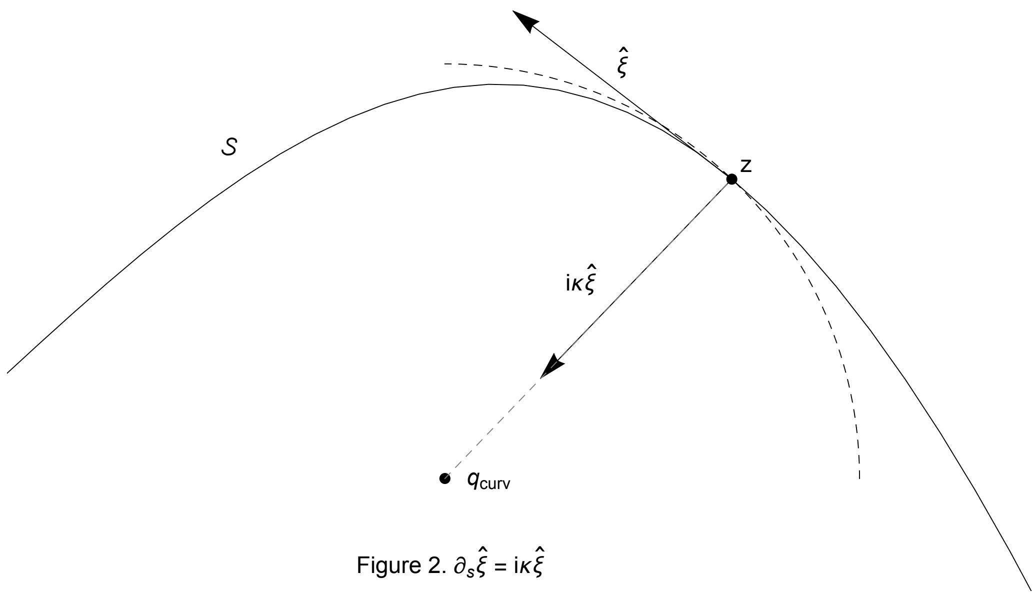


Figure 2. $\partial_s\hat{\xi} = i\kappa\hat{\xi}$

*(ii) Show that $\partial_s \hat{\xi} = i\kappa\hat{\xi}$ and $\partial_s \Psi = \hat{\xi} \bullet (i\mathcal{K})$.*

The direction of the tangent $\hat{\xi}$ is always orthogonal to a ray from the centre of curvature q to z (Figure 2). Since $\hat{\xi}$ is a unit vector tangent to the curve S at z, the change in $\hat{\xi}$ in the direction of $\hat{\xi}$ is zero, but in moving on a unit of S, it does rotate in the direction of the direction of (i$\hat{\xi}$) towards the centre of curvature by an angle that is equal to 1/r, which is the curvature $\kappa$ at z, where r is the radius of curvature.

We can immediately write $\partial_s \hat{\xi} = i\kappa\hat{\xi}$, but I would like to suggest a likeness to (rotation of) the gradient $\nabla$: Let "T" and "R" be orthogonal axes intersecting on the curve S at z, where axis T is parallel to $\hat{\xi}$ and axis R has the direction of (i$\hat{\xi}$) towards the centre of curvature. We can write $\partial_s =$ $(\partial_T + i\,\partial_R)$, so that

$$\partial_s\hat{\xi} = (\partial_T + i\,\partial_R)\,\hat{\xi} = (0 + \text{i k}\,\hat{\xi}) = \text{i}\kappa\hat{\xi}$$

as was to be shown.

*Show that $\partial_s \Psi = \hat{\xi} \bullet (i\mathcal{K})$.*

By (23), p. 497, *The direction of $\nabla\Psi$ is orthogonal to the streamlines,* and it is "*the direction along which $\Psi$ increases*." We are somewhat confused as to whether $\Psi$ is the stream function of $\mathcal{K}$, as in the problem statement, or the stream function of f, as implied in Figure [19]. Since $\nabla\Psi$ is orthogonal to $\mathcal{K}$, then $\hat{\xi} \bullet (\mathrm{i}\mathcal{K})$ represents $\nabla\Psi$ on $\hat{\xi}$. That is, $\nabla\Psi$ is the vector pointing in the direction along which $\Psi$ at f(z) increases with movement on S. To restate this as a derivation:

$$\partial_s \Psi = \hat{\xi} \bullet \nabla\Psi = \hat{\xi} \bullet i(-i\nabla\Psi) = \hat{\xi} \bullet (i\mathcal{K})$$

*(iii) Deduce that*

$$\partial_{\tilde{s}} \tilde{\kappa} = \Psi^2 \partial_s \kappa + \Psi\, (\partial_s \mathcal{K}) \bullet \hat{\xi}.$$

Begin with the result of (i):

$$\partial_{\tilde{s}} \tilde{\kappa} = \Psi\, \partial_s \left[\kappa\Psi + \mathcal{K} \bullet \hat{\xi}\right]$$

$$= \Psi \left[\partial_s(\kappa\Psi) + \partial_s\left(\mathcal{K} \bullet \hat{\xi}\right)\right] \qquad \text{linearity of } \partial_s$$

$$= \Psi \left[((\partial_s\kappa)\,\Psi + \kappa(\partial_s\Psi)) + \left((\partial_s\mathcal{K}) \bullet \hat{\xi}\right) + \mathcal{K} \bullet \partial_s\hat{\xi}\right] \text{ chain rule (works with inner product)}$$

$$= \Psi \left[(\partial_s\kappa)\,\Psi + \kappa\left(\hat{\xi} \bullet \mathrm{i}\mathcal{K}\right) + (\partial_s\mathcal{K}) \bullet \hat{\xi} + \mathcal{K} \bullet \mathrm{i}\kappa\,\hat{\xi}\right] \qquad \text{two substitutions from (ii)}$$

$$= \Psi \left[(\partial_s\kappa)\,\Psi + \left((\partial_s\mathcal{K}) \bullet \hat{\xi} + \kappa\left((\mathrm{i}\mathcal{K}) \bullet \hat{\xi}\right) + \kappa\left(\mathcal{K} \bullet \left(i\,\hat{\xi}\right)\right)\right)\right] \text{ rearrange}$$

$$= \Psi^2 \partial_s \kappa + \Psi\, (\partial_s \mathcal{K}) \bullet \hat{\xi} \qquad \text{use hint and multiply through by } \Psi$$

This is what we wanted.

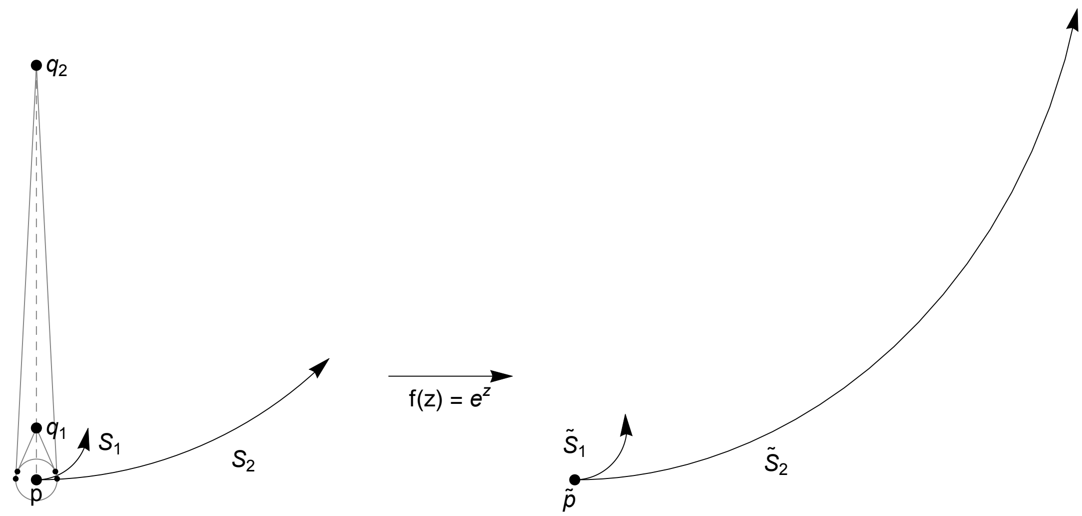


Figure 3. Conformal angles under f = $e^z$

*(iv) Recall Ex. 18, p. 261, in which we saw how Newton attempted to define the "angle" $\Theta$ between two touching curves as the difference of their curvatures: $\Theta \equiv (\kappa_1 - \kappa_2)$. Although this is not quite conformally invariant, show that $\left[\Theta^2 \big/ \partial_s \Theta\right]$ is conformally invariant. ... Kasner's Invariant.*

Figure 3 represents the geometry of the problem where f(z) = $e^z$. The small disc D on p is the disc described on p. 261, part (ii). It turns out to be not very useful in this proof, but it may be helpful in understanding it. I believe that $\Theta$ can be understood as the angle $2\epsilon$D. The equation for $\Theta$ used in this proof is that introduced by Newton: $\Theta \equiv \kappa_1 - \kappa_2$. If $\left[\Theta^2 \big/ \partial_s \Theta\right]$ is conformally invariant, then we must show that $\left[\tilde{\Theta}^2 \big/ \partial_s \tilde{\Theta}\right] = \left[\Theta^2 \big/ \partial_s \Theta\right]$.

Using the result from (iii), we note that $\Psi$ and $\mathcal{K}$ depend only on f at p, which is the same for both curves. Hence they never take diacriticals or subscripts in this problem. $\hat{\xi}$ is a unit vector tangent to a curve, but Vasco has pointed out that there is no need to distinguish between $\hat{\xi}_1$ and $\hat{\xi}_2$ because "the two curves touch and therefore **do not intersect**" (personal communication, Forum) at p. Note that $(\partial_{s_1} \kappa_1 - \partial_{s_2} \kappa_2) = (\partial_s \kappa_1 - \partial_s \kappa_2) = \partial_s (\kappa_1 - \kappa_2)$. The partials $\partial_{s_1}$ and $\partial_{s_2}$ resolve to $\partial_s$ because the distance to be traversed is infinitesimal on both curves and they point in the same direction at p. The same is true of $\partial_{\tilde{s}_1} \tilde{\kappa}_1 - \partial_{\tilde{s}_2} \tilde{\kappa}_2$ at $\tilde{p}$, because f(z) is analytic. Using the result of part (i), Exercise 18, p. 21, we find that $\tilde{\Theta} = \Theta/|f'(p)| = \Theta\Psi$, so $\Psi^2 = \frac{\tilde{\Theta}^2}{\Theta^2}$.

$$\partial_{\tilde{s}} \tilde{\kappa}_1 = \Psi^2 \partial_s \kappa_1 + \Psi\, (\partial_s \mathcal{K}) \bullet \hat{\xi} \qquad \text{(from (iii))}$$

$$\partial_{\tilde{s}} \tilde{\kappa}_2 = \Psi^2 \partial_s \kappa_2 + \Psi\, (\partial_s \mathcal{K}) \bullet \hat{\xi}$$

$$\partial_{\tilde{s}} \tilde{\kappa}_1 - \partial_{\tilde{s}} \tilde{\kappa}_2 = \left(\Psi^2 \partial_s \kappa_1 - \Psi^2 \partial_s \kappa_2\right) + \left(\Psi\, (\partial_s \mathcal{K}) \bullet \hat{\xi} - \Psi\, (\partial_s \mathcal{K}) \bullet \hat{\xi}\right) \quad \text{(substract 2nd line from 1st)}$$

$$\partial_{\tilde{s}}(\tilde{\kappa}_1 - \tilde{\kappa}_2) = \Psi^2\, \partial_s(\kappa_1 - \kappa_2)$$

$$\partial_{\tilde{s}}\tilde{\Theta} = \Psi^2\, \partial_s\Theta \qquad \text{(substitute for difference in } \kappa\text{)}$$

$$\partial_{\tilde{s}}\tilde{\Theta} = \frac{\tilde{\Theta}^2}{\Theta^2}\, \partial_s\Theta \qquad \text{(substitute for } \Psi^2\text{)}$$

$$\frac{\tilde{\Theta}^2}{\partial_{\tilde{s}}\tilde{\Theta}} = \frac{\Theta^2}{\partial_s\Theta} \qquad \text{(rearrange)}$$

This is what we set out to show.