Needham, Chapter 12, Exercise 1, p. 505-506.
G. Palmer, July 29, 2018, 10:25 pm PST

Out[2526]=

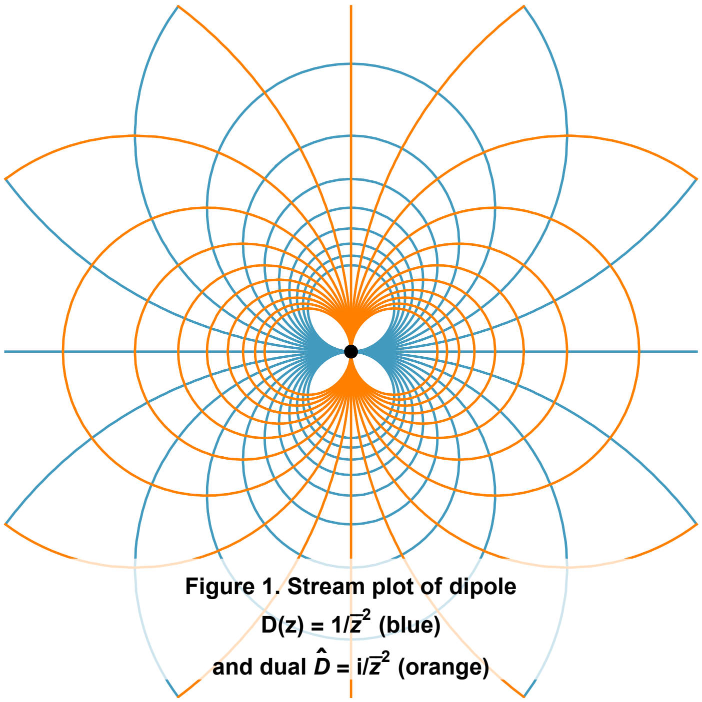

Figure 1. Stream plot of dipole
$D(z) = 1/\overline{z}^2$ (blue)
and dual $\hat{D} = i/\overline{z}^2$ (orange)

1 (i) Show that the dual of a dipole is a dipole.

The second equation on p. 491 provides a definition of a dipole, which is the coalescence of a source and a sink.

$$D(z) = \frac{ke^{-i\phi}}{\overline{z}^2}$$

where k is a real number and ϕ is the angle of the axis, the line through the centres of the doublet that creates the dipole.

Needham notes that *If at each point we rotate* [a Pólya vector field] $\overline{H}$ *through a fixed angle* ϑ *then we obtain the Pólya vector field of the analytic function* $H_\vartheta \equiv e^{-i\vartheta} H$*, namely* $\overline{H}_\vartheta = e^{i\vartheta}\,\overline{H}$*. Thus, this rotated vector field* [$\overline{H}_\vartheta$] *is automatically sourceless and irrotational, and its complex potential is* $\Omega_\vartheta = e^{-i\vartheta}\Omega$. When $e^{i\vartheta}\,\overline{H} = i\overline{H}$, it is called by Needham the *dual of* $\overline{H}$. He refers to the potential and stream functions of the dual flow as the *dual potential* and the *dual stream function* (p. 508).

Let $D(z) = \overline{H} = \frac{ke^{-i\phi}}{\overline{z}^2}$. Then D(z) has the dual $\frac{ike^{-i\phi}}{\overline{z}^2} = \frac{ke^{i(-\phi+\pi/2)}}{\overline{z}^2}$. This is just a rotation of the field vector D by $\pi/2$. It is clear that the dual of a dipole is a dipole because ($-\phi + \pi/2$) may be any real number.

In Figure 1, the phase portrait of the field vector D is shown in blue. The phase portrait of its dual $\hat{D}$ is shown in orange.

Out[2619]=

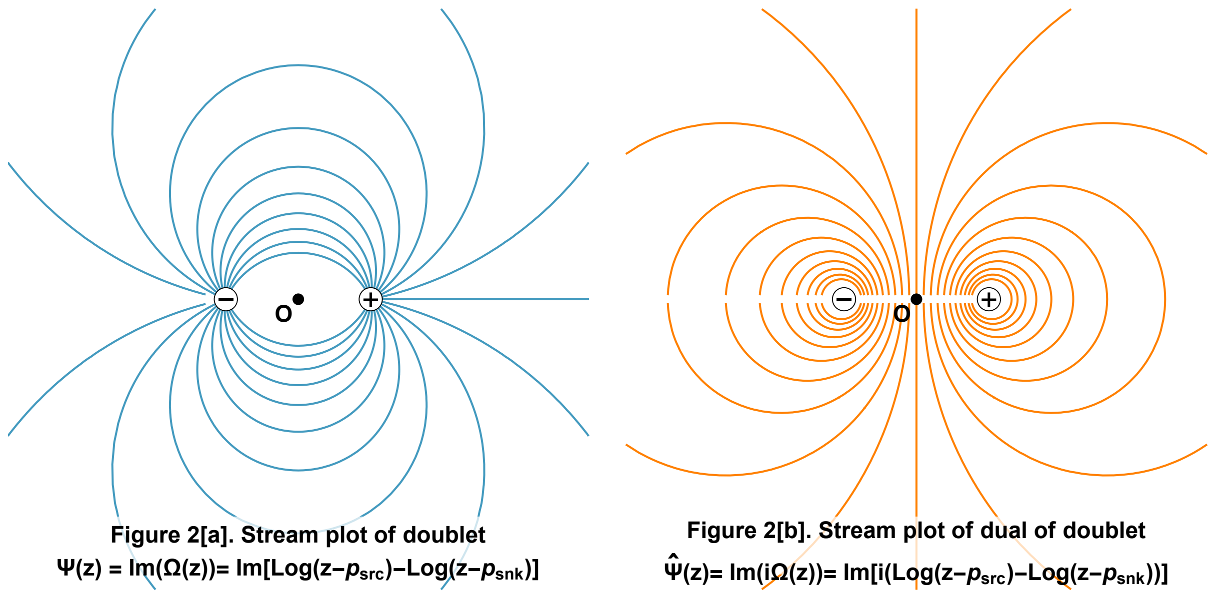

Figure 2[a]. Stream plot of doublet
$\Psi(z) = \mathrm{Im}(\Omega(z))= \mathrm{Im}[\mathrm{Log}(z-p_{src})-\mathrm{Log}(z-p_{snk})]$

Figure 2[b]. Stream plot of dual of doublet
$\hat{\Psi}(z)= \mathrm{Im}(i\Omega(z))= \mathrm{Im}[i(\mathrm{Log}(z-p_{src})-\mathrm{Log}(z-p_{snk}))]$

(ii) Think of a dipole as the limiting form of a doublet [a source and sink of equal strength]. Sketch the dual of a doublet, and thereby make geometric sense of part (i).

Streamlines connecting the poles of a doublet are segments of circles (p. 453). When the source and sink points are drawn towards one another along the axis (horizontal in 1 and 2[a]), segments of streamlines gain curvature, but they remain segments of perfect circles (p. 490), so it must be the case that the streamline circles shrink in radius. As the poles become very near to each other, the originating doublet streamlines morph into dipole streamlines, which are also circles, as we have shown in various exercises.

The dual of the doublet (Figure 2b) results from a different process than the dual of the dipole. As indicated in Figure [1], p. 509, a vortex occurs where the original source is rotated by $\vartheta = \pi/2$. The dual of a source is a vortex, as shown in Figure 2[b]. The dual of a sink is also a vortex but of opposite sign. When the centres of the dual vortices are brought very close together, the result is identical to the dual of the dipole.Thus, one can create the dual of a dipole in either of two ways: (1) rotate the dipole by $\pi/2$; (2) rotate a source and a sink independently by $\pi/2$ to make two vortices and let the two centres converge to a dipole.