

Notes on finding and plotting streamlines and potentials

G. Palmer, September 6, 2018, 12:10 noon PDT

Streamlines are mentioned in chapter 10 and considered extensively in chapter 11 in conjunction with phase portraits, yet Needham's explanations left me wondering how to calculate the streamlines as curves. The purpose of these notes is to explain how to calculate streamlines and lines of potential given either a vector field or the complex potential of a vector field.

Notation

Vector fields (flows, force fields) may be designated as H or X , as in the example $1/z$, or as a Pólya field $\overline{H}$, as in $i/(\overline{z}-\overline{p})$. Vasco (p.c., 10/6/18) has noted that "if a vector field is the Pólya vector field of an analytic function," then this Pólya vector field is sourceless and conservative (irrotational). But it may nevertheless have a stream function ψ , a potential function ϕ , or both.

"For example take the case of the vector field $x + iy$, which is an analytic function. The Pólya vector field is $\overline{z}$, a non-analytic function. For z we have $P = x$ and $Q = y$ and so div is 2 and curl is zero. So the analytic vector field z is *not* sourceless *and* conservative. For $\overline{z}$ we have $P = X$ and $Q = -y$ and so div is zero and curl is zero. So $\overline{z}$, the Pólya vector field of an analytic function z , is sourceless and conservative."

Note that it is div (flux density) and curl (work density), not flux and work, that provide the criteria for determining whether a function is sourceless and/or conservative. See also, p. 483, last paragraph. But flux and work may also be the criteria:

$$F[X, \text{any closed loop}] = 0 \quad \text{and} \quad W[X, \text{any closed loop}] = 0 \quad (\text{p. 476, (2)})$$

The complex potential $\Omega = \phi + i\psi$. These stream and potential functions ϕ and ψ correspond to $W[\overline{H}, K]$ and $F[\overline{H}, K]$ and it follows that $\int_K H(z) dz$ corresponds to $\Omega(z)$. (p. 483, (11))

To find work and flux, one may write

$$W[X, K] = \int_K X \cdot T ds \quad (1) \quad (\text{p. 474})$$

$$F[X, K] = \int_K X \cdot N ds \quad (2) \quad (\text{p. 473})$$

This gives the work and flux of any vector field X , but $W[X, K]$ and $F[X, K]$ are not the same as ϕ or ψ , which are the work and flux calculated on the Pólya field (p. 495).

$$\int_K H(z) dz = W[\overline{H}, K] + i F[\overline{H}, K] \quad (3) \quad (\text{p. 483})$$

There is some room for confusion here in that (1) and (2) at first appear to be contradicted by (3), because in (1), (2) and (3), both X and H indicate field vectors and the right hand sides of (1) and (2) have X in their arguments, yet in (3) the field vector H requires the argument $\overline{H}$ on the RHS.

Apparently “X” may stand for “either H or $\overline{H}$ ”. One would not expect to see “ $\overline{X}$ ” written. So $\int_K H(z) dz$ is given by the work and the flux of the Pólya vector field. To paraphrase Vasco, if $\overline{H}$ is sourceless, meaning that $\nabla \cdot \overline{H} = 0$, and $F[\overline{H}, K] = 0$ on any closed loop, then $\psi = F$; and if $\overline{H}$ is conservative, meaning that $\nabla \times \overline{H} = 0$, and $W[\overline{H}, K] = 0$ on any closed loop, then $\phi = W$. F and W are calculated as integrals on contours that are crossed by streamlines (in the case of F) or that parallel streamlines (in the case of W). If $\overline{H}$ is both sourceless and conservative, we know that H is analytic, but H may not be sourceless and conservative (p. 483 (12)).

Calculating ψ from flux on a contour

$$\psi(z) = F[\overline{H}, K] \quad (\text{p. 495})$$

We are given the example

$$\overline{H} = -iz \quad \equiv \quad \overline{H} = \begin{pmatrix} y \\ -x \end{pmatrix} \quad (\text{p. 496})$$

In complex analysis, the format $\mathbf{X} = \begin{pmatrix} u \\ v \end{pmatrix}$ implies $X = u + iv$, where u and v are real numbers.

The lines of flow of $-iz$ are circular and clockwise, so they cross the real axis with a direction that can be described as

$$\mathbf{N} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

The real axis is the contour of integration, which was chosen for convenience, as it is described by a single real variable x, which allows integration in either direction with no effect on the result. Since we can let $ds = dx$, there is no need to parameterize. Then, by (2), above,

$$\psi = F[\overline{H}, K] = \int_K \overline{H} \cdot \mathbf{N} ds = \int_0^R \begin{pmatrix} y \\ -x \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} dx = \int_0^R x dx = \frac{R^2}{2}$$

The value of R on x corresponds to the radius of each circular streamline. See p. 496, figure [18].

Calculating ϕ and ψ from the gradient by integrating the real and imaginary parts of $\overline{H}$

If a function is analytic, it's Pólya field is conservative (no curl, irrotational, no work on closed loop) and sourceless (no div, no flux on closed loop), but nevertheless may possess a potential ϕ and/or a stream function ψ ! Given H or $\overline{H}$, these may be calculated as above, or by using knowledge of the gradient ∇ applied to ϕ and ψ .

$$\overline{H} = \nabla\phi = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \phi = \begin{pmatrix} \partial_x\phi \\ \partial_y\phi \end{pmatrix} \quad (\text{p. 500})$$

$$\overline{H} = -i\nabla\psi = \begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix} \psi = \begin{pmatrix} \partial_y\psi \\ -\partial_x\psi \end{pmatrix} \quad (\text{p. 498})$$

It follows that if $\overline{H} = \begin{pmatrix} P(x, y) \\ Q(x, y) \end{pmatrix}$

$$\phi = \int P \, dx = \int Q \, dy \quad (\text{because } \int (\partial_x\phi) \, dx = \phi \text{ and } \int \partial_y\phi \, dy = \phi)$$

$$\psi = \int P \, dy = -\int Q \, dx$$

ϕ and ψ may include a constant of integration.

Calculating ϕ , ψ , and H from Ω

If Ω is given, then $\phi = \text{Re}(\Omega)$ and $\psi = \text{Im}(\Omega)$. $H = \Omega'$.