

p. 497. Knowing the stream function we are now in a position to draw the special phase portrait. Choosing  $k = (1/2)$  we find that the radii of the streamlines are  $\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots$

On p. 495, ¶ 2 it is given that  $\Psi = 0, \pm k, \pm 2k, \pm 3k, \dots$ , as in [17], and on p. 497 we find that  $\Psi = \frac{1}{2} R^2$ . Hence,

$$\frac{1}{2} R^2 = 0, \pm k, \pm 2k, \pm 3k, \dots$$

$$R^2 = 0, \pm 1, \pm 2, \pm 3, \dots$$

$$R = 0, \sqrt{1}, \sqrt{2}, \sqrt{3}, \dots$$

**P. 503 (1).** Find  $\Omega^{-1}$ , where  $\Omega = -(1/z) + c$ .

Vasco has provided the answer in personal communications. My version is as follows:

Decompose omega into a series of transformations:

$z \rightarrow 1/z$  results in  $1/z$   
 $z \rightarrow -z$  results in  $-1/z$   
 $z \rightarrow z + c$  results in  $-1/z + c$

Reverse the series to get  $\omega^{-1}$

$z \rightarrow z - c$  results in  $z - c$   
 $z \rightarrow -z$  results in  $-z + c$   
 $z \rightarrow 1/z$  results in  $1/(-z + c) = -1/(z - c)$

or

$$\omega(\omega^{-1}) = \omega(-1/(z - c)) = z$$

$$-1/(z - c) \rightarrow -(z - c) = -z + c$$

$$-z + c \rightarrow z - c$$

$$z - c \rightarrow z$$

**P. 503 (2).** A uniform eastward flow has complex potential  $\Omega = z$ . If we insert a dipole of complex potential  $\Omega = (1/z)$  into this flow then the new flow will be the superposition of the two individual flows and thus will have complex potential

$$\Omega(z) = z + \frac{1}{z}.$$

Using your computer, you may now verify that the streamlines and equipotentials are as shown in [22]. Note how the streamlines emanating from the dipole are deformed out of perfect circularity by the uniform flow, but this distortion diminishes as the origin is approached.

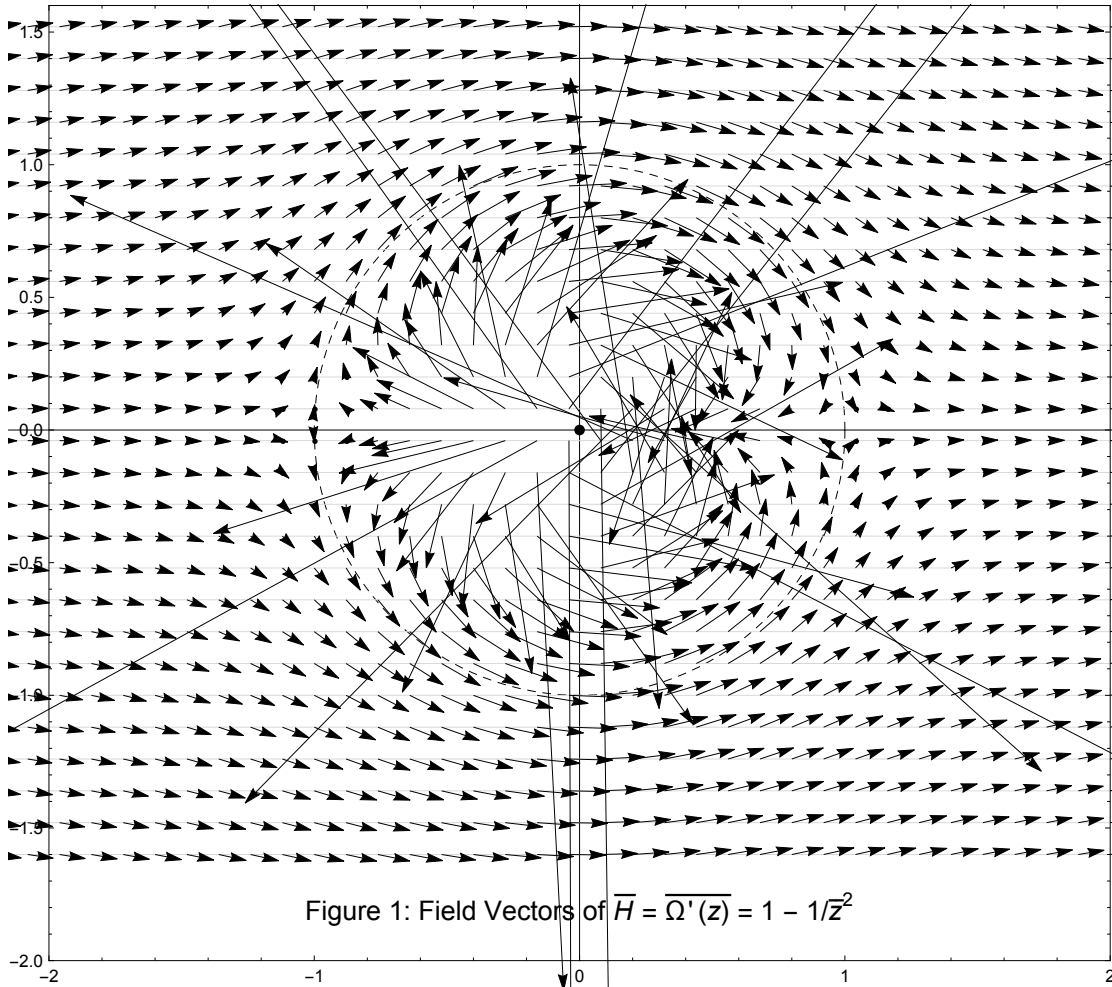


Figure 1 shows Pólya field vectors that have been inferred from  $\Omega(z) = z + 1/z$ , a complex potential that represents the composition of uniform eastward flow on which a dipole  $1/z$  is superimposed. A close look and a bit of imagination reveals streamlines similar to those in [22], p. 503. Streamlines are normally inferred from  $H$  or  $\bar{H}$ , where  $H = \Omega'$  (p. 502). Streamlines are curves tangent to the slope of  $H$  or  $\bar{H}$ . They are described by antiderivatives of the slope of  $H$  or  $\bar{H}$ . We know from (1) that the streamlines of the dipole with potential  $1/z$  are likely to be circles that intersect with the origin. Streamlines of the continuous eastward flow must be equally spaced and point in the direction of the flow. When they are combined, the result should be a copy of [22]. We attempted to follow the method described on p. 496. First, rewrite  $\bar{H}$  as a vector:

$$H = \Omega' = 1 + 1/z^2$$

$$\begin{aligned}
\overline{H} &= 1 - 1/\bar{z}^2 = (\bar{z}^2 - 1)/\bar{z}^2 = z^2(\bar{z}^2 - 1)/z^2 \bar{z}^2 \\
&= [(z\bar{z})^2 - z^2]/(z\bar{z})^2 = (|z|^4 - z^2)/|z|^4 \\
&= 1 - z^2/|z|^4 \\
&= 1 - (x^2 - y^2)/\left((x^2 + y^2)^2\right) - 2ixy/\left((x^2 + y^2)^2\right) \\
&= \begin{pmatrix} 1 - (x^2 - y^2)/(x^2 + y^2)^2 \\ -2xy/(x^2 + y^2)^2 \end{pmatrix}
\end{aligned}$$

At this point I failed to fully understand the problem. I did not see a way to choose  $K$ , and so I based  $N$  on the streamlines, which is wrong.  $K$  is any line crossed by the streamlines (p. 501, [21]). Then one can calculate the flux (stream function)  $\psi$  crossing a line connecting any two streamlines. One may choose any line connecting two streamlines, because the streamlines have constant flux. If we choose a vertical line it can connect two horizontal streamlines or two circular streamlines, or it can connect a circular streamline to a horizontal streamline. Then, it is convenient to write  $N = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , because it is normal to  $K$ , and  $ds = dy$  to indicate integration on the vertical line  $K$ .

$$\begin{aligned}
\psi &= \int_K (\overline{H} \cdot N) ds = \int_K \begin{pmatrix} 1 - (x^2 - y^2)/(x^2 + y^2)^2 \\ -2xy/(x^2 + y^2)^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dy \\
&= \int_0^Y \left[ 1 - (x^2 - y^2)/(x^2 + y^2)^2 \right] dy \\
&= \int_0^Y dy - \int_0^Y (x^2 - y^2)/(x^2 + y^2)^2 dy \\
&= Y - \frac{Y}{x^2 + Y^2} \implies \psi = y - \frac{y}{x^2 + y^2}
\end{aligned}$$

Rearrange, letting  $r^2 = x^2 + y^2$ .

$$\psi r^2 = y r^2 - y$$

$$\psi x^2 + \psi y^2 = y x^2 + y^3 - y$$

$$y^3 - \psi y^2 + yx^2 - y = \psi x^2$$

$$y(y^2 - \psi y + x^2 - 1) - \psi x^2 = 0$$

When  $\psi = 0$ , this becomes  $y(y^2 + x^2 - 1) = 0$ . Then,  $y = 0$ , or  $y^2 + x^2 = 1$ , which are equations for the real line and the circle. Vasco wrote: “Now set  $\psi = [-1.8, -1.6, -1.4, \dots, -0.2, 0, 0.2, \dots, 1.8]$  and draw the resulting  $x, y$  curves to obtain figure [22]. He doesn’t say why he chose these particular values for the flux, but in Figure 2, they work well using the scale of Figure 1. The upper and lower bounds of  $x$  and  $y$  must be equal in order to get a perfect circle in *Mathematica*’s function `ContourPlot[]`. The following function was used to plot the  $\psi$  lines:

```
phiPlot[phi_, const_] := ContourPlot[(x/(x^2 + y^2) + x + const) == phi,
  {x, -bnds, bnds}, {y, -bnds, bnds}, ContourStyle -> {Dashed, Gray};
```

The potential function or work is  $\phi(z) = W[\bar{H}, K] = \int_K \bar{H} \cdot T \, ds$ , where  $T = -iN = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ , and it should still be true that  $ds = dy$ . Then the work along  $K$  is

$$\begin{aligned} \phi &= \int_K (\bar{H} \cdot T) \, ds = \int_K \begin{pmatrix} 1 - (x^2 - y^2) / (x^2 + y^2)^2 \\ -2xy / (x^2 + y^2)^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} ds \\ &= 2 \int_0^y xy / (x^2 + y^2)^2 dy \\ &= -\frac{x}{x^2 + y^2} \end{aligned}$$

This did not plot as lines orthogonal to the streamlines. On p. 502 we are given the vector equation that represents the CR equations for  $\Omega$ . We know that  $\Omega$  is analytic, because it is composed of the analytic functions  $z$  and  $1/z$ . From the first CR equation, calculate

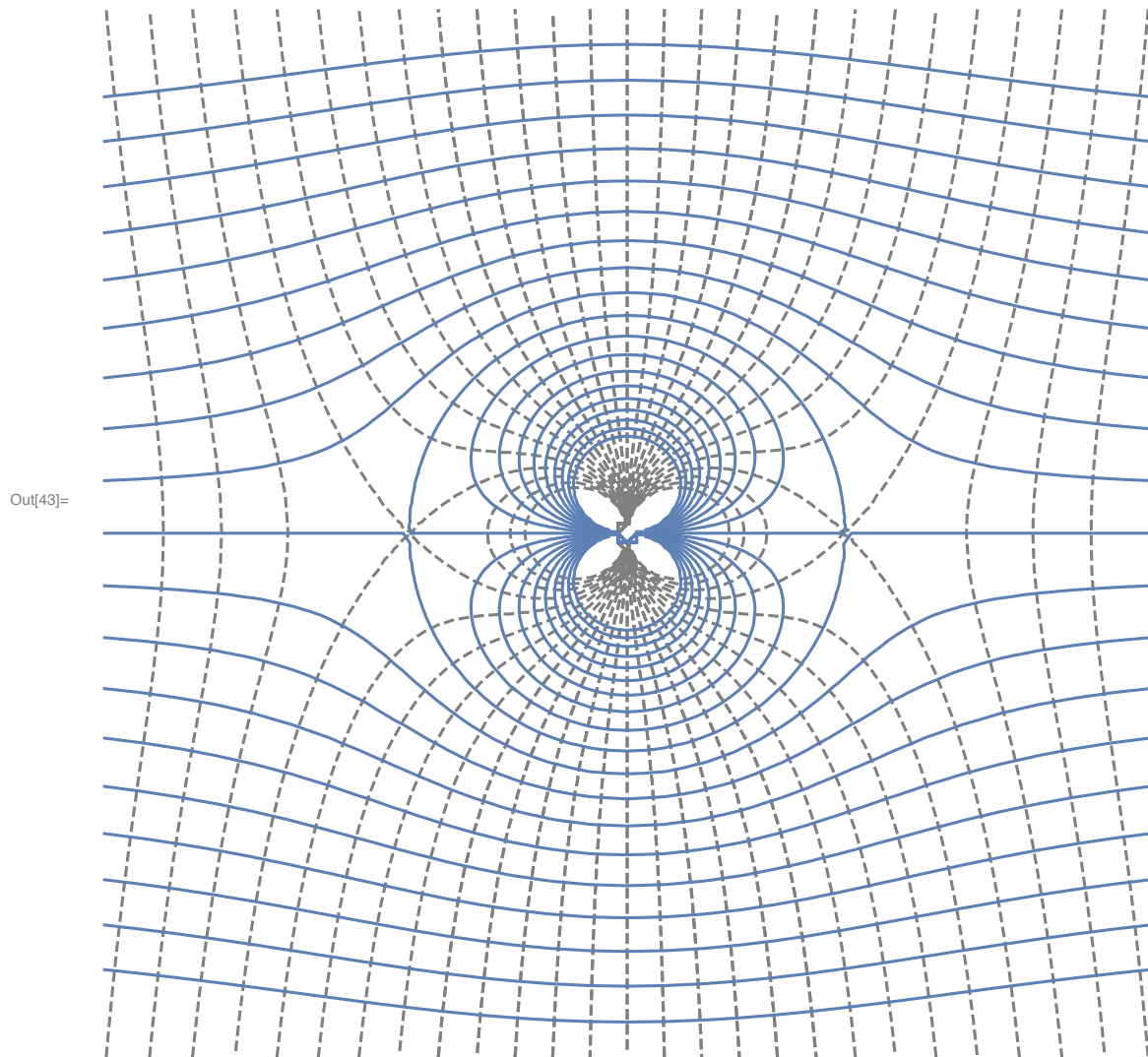
$$\frac{\partial}{\partial x} \phi = \frac{\partial}{\partial y} \psi = \frac{\partial}{\partial y} \left( y - \frac{y}{x^2 + y^2} \right) = \frac{2y^2}{(x^2 + y^2)^2} - \frac{1}{x^2 + y^2} + 1$$

Then

$$\phi = \int \frac{\partial}{\partial x} \phi \, dx = \frac{x}{x^2 + y^2} + x + C$$

Letting  $C = 0$ , this equation plots the dashed gray lines of potential in Figure 2b. This figure appears to reproduce [22] except that the potential lines appear to begin at the origin and the streamlines are not plotted with arrows. The plotting function is `ContourPlot[( $\frac{x}{x^2 + y^2} + x + c$ ) ==  $\phi$ , {x, 0, 2.4}, {y, 0, 2.4}]`. Oddly, the potential lines at larger absolute values of  $x$  could only be obtained by altering the constant  $c$ . Alterations to  $\phi$  had no effect.

Figure 2. Contour plot of stream function  $\psi$  (blue) and potential function  $\phi$  (dashed, gray) for  $\Omega = z + 1/z$



Note that *Mathematica* also has a built in function `StreamPlot` to plot the streamlines directly from  $\overline{H}$ . It is interesting to compare results. The circular flow ( $x^2 + y^2 = 1$ ) is not plotted. The circles of the bipole are distorted and fail to touch the real line.