

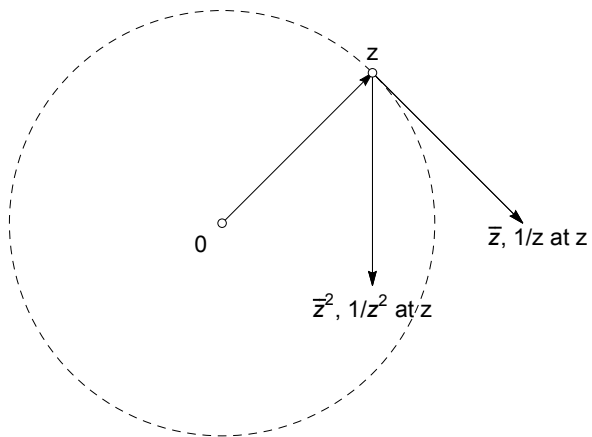


Figure 6. $\overline{z^n} = 1/z^n$

P. 482.

H is the flow vector attached to z . $\overline{H}$ is the conjugate of the flow vector, also attached to z (Figure 6).

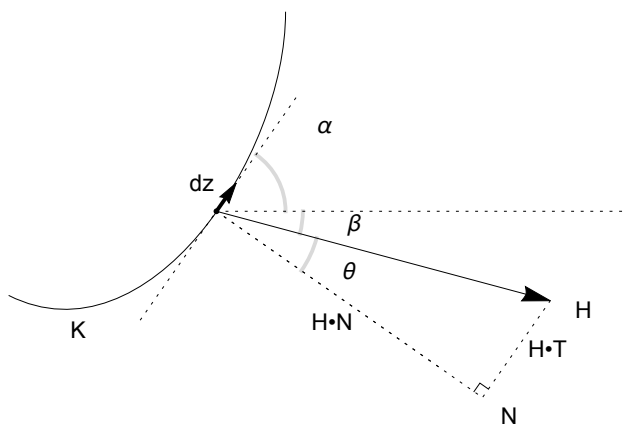


Figure 6b. Calculations with H compared to Pólya field

p. 482. The Great advantage of the Pólya vector field is that the angle θ that it makes with the contour (see [7]) is given by $\theta = \alpha - (-\beta)$

The purpose of this note is to test Needham's assertion that the Pólya vector field has a great advantage over the corresponding non-Pólya vector field (pp. 482-483). This figure corresponds to Figure [7], p. 482, but it presents H rather than $\overline{H}$. I call it Figure 6b as it was inserted between 6 and 7. Let dz be an infinitesimal tangent to K . Figure N shows that $H dz = |H| e^{i(\beta + \alpha)}$, where $\alpha = \arg(dz)$, $\beta = \arg(H)$, and θ is the angle that H makes with N .

$$H dz = |H| e^{i(\beta + \alpha)} ds = |H| [\cos(\beta + \alpha) + i \sin(\beta + \alpha)] ds$$

We can write

$$\begin{aligned} H \cdot N &= |H| \cos \theta \\ H \cdot T &= |H| \sin \theta \end{aligned}$$

But what is θ ? It is not $(\beta + \alpha)$ as it is for the Pólya vector.

$$\theta = \beta - \alpha + \pi/2 = \arg(H) + \arg(dz) + \pi/2.$$

We can't write

$$H dz = |H| e^{i\theta} ds = |H| [\cos \theta + i \sin \theta] ds$$

because $\theta \neq (\beta + \alpha)$. We can write (pp. 473-474)

$$\begin{aligned} W[H, K] &= \int_K (H \cdot T) ds \\ F[H, K] &= \int_K (H \cdot N) ds \end{aligned}$$

But we can't write

$$\int_K H(z) dz = W[H, K] + i F[H, K]$$

because the angles used in calculating $W[H, K]$ and $F[H, K]$ are wrong: $\theta \neq (\arg(H) + \arg(dz))$.

I conclude that Needham is correct. The Pólya field provides a more direct way to visualize $H dz$ and calculate integrals.

The Pólya vector field is relative: Some evidence.

Are some functions Pólya and others not? Here is some evidence from the chapter that the Pólya vector field is relative to whatever vector field one chooses.

p. 481. It is given that $H = |H| e^{i\beta}$, but β is unrestricted as to sign. In II, 1, paragraph 2, we have

$$\overline{H(z)} = |H| e^{-i\beta}.$$

p. 483. "a computer can instantly draw the Pólya vector field of *any function you wish to integrate*" [emphasis added]. The obvious implication is that any function can have a PVF. For example, consider the PVF of $\bar{z}^2 z$:

$$\begin{aligned} H &= \bar{z}^2 z \\ \overline{H} &= z |z|^2 \end{aligned}$$

p. 484. We find that the PVF of $H(z) = \bar{z}$ is $\overline{H(z)} = z$.

p. 496. If $H(z) = i\bar{z}$, $\overline{H(z)} = -iz$.

p. 503 (3) "a source of strength 2π at the origin has a vector field $\overline{H} = (1/\bar{z})$." Here there is no explicit designation of the vector field as Pólya, but " $\overline{H}$ " is defined as a PVF. It follows, too, that

$$H(z) = 1/z$$

$$\int (\Omega' = H) dz = \ln|z| + i\theta_L(z)$$

From these bits, it appears that the Pólya vector field is relative to whatever function one chooses, and one may choose functions that may or may not contain conjugates. However, it may be that it works better to choose certain kinds of functions to be H and others to be $\overline{H}$.

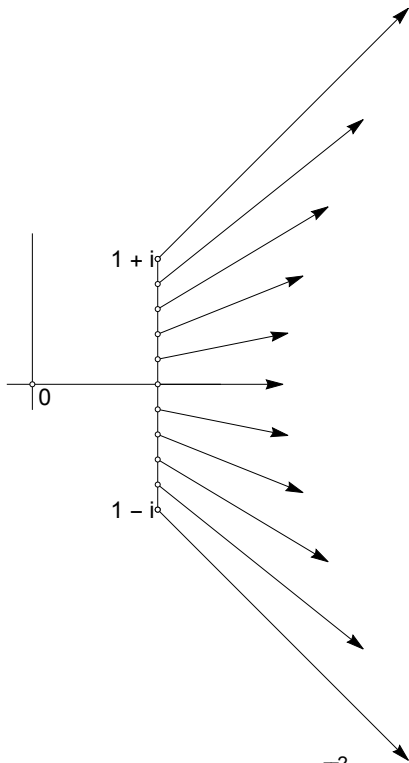


Figure 6c. Pólya vector field of $\bar{z}^2 z$
on $1-i$ to $1+i$

p. 483 *the integral of $(\bar{z}^2 z)$ along the line segment from $1-i$ to $1+i$ is clearly a positive multiple of i . Why?*

The Pólya field is symmetrical on the real axis (Figure 6c), so all the curl above the axis is exactly canceled by curl below the axis. Hence there is no work, and $W[\bar{H}, K] = 0$. Since $F[\bar{H}, K]$ is a real number, the integral $\int_K H(z) dz = W[\bar{H}, K] + iF[\bar{H}, K]$ must be a multiple of i . Since the flow is all from left to right, it must be a positive multiple of i , so $\int_K H(z) dz$ must also be a positive multiple of i .

P. 484. $H(z) = z$, ... flows radially outwards from the origin, like a source.

How do we know this? Perhaps because the vector field is everywhere tangent to the streamlines (p. 453). Then a streamline at z must have the same direction as z and the flow points away from the origin.

P. 485. But the Pólya vector field $(1/\bar{z})$ is an old friend—it is a source of strength 2π located at the origin.

I don't find a previous statement of this, but on p. 455, we find

$$V(z) = \frac{S}{2\pi} \left(\frac{1}{\bar{z}} \right),$$

where S is “the amount of fluid being pumped in at 0,” i.e. the strength.

By substituting 2π for S, we get $V(z) = 1/\bar{z}$.

We note that the quality of being a source belongs to the vector field. It is not something assigned to the vector field post hoc. One could decide to create a source, but then one would have to find an appropriate expression for a vector field to calculate with it. Note also that this particular $V(z)$ is symmetric (p. 454 and [4a], p. 455).

P. 487 (17) Turning to the Pólya vector field itself, and substituting (13), we find

$$\overline{H(z)} = \overline{H(0)} + (\nabla \cdot \overline{H}) \frac{z}{2} + (\nabla \times \overline{H}) \frac{iz}{2} + C\bar{z}.$$

The derivation is easy, but managing the signs and conjugates is tricky, particularly with C.

We are given (13): $i \partial_x H - \partial_y H = \nabla \times \overline{H} + i \nabla \cdot \overline{H}$.

$$H(z) - H(0) = \frac{1}{2} [\partial_x H - i \partial_y H] z + \frac{1}{2} [\partial_x H + i \partial_y H] \bar{z}$$

$$= \overline{C}z + \frac{1}{2} [-i \partial_x H + \partial_y H] i \bar{z} \quad \overline{C} = \frac{1}{2} [\partial_x H - i \partial_y H]$$

$$H(z) = H(0) + \frac{1}{2} [-(\nabla \times \overline{H} + i \nabla \cdot \overline{H})] i \bar{z} + \overline{C}z \text{ by substitution from (13) \& rearrange}$$

$$= H(0) + \frac{1}{2} [-i \nabla \times \overline{H} + \nabla \cdot \overline{H}] \bar{z} + \overline{C}z \text{ move } i \text{ from outside to inside brackets}$$

$$\overline{H(z)} = \overline{H(0)} + \overline{\left[-i \nabla \times \overline{H} + \nabla \cdot \overline{H} \right] \frac{\bar{z}}{2}} + C\bar{z} \text{ conjugate both sides}$$

$$= \overline{H(0)} + [\nabla \cdot \overline{H} + i \nabla \times \overline{H}] \frac{z}{2} + C\bar{z}$$

$$= \overline{H(0)} + (\nabla \cdot \overline{H}) \frac{z}{2} + (\nabla \times \overline{H}) \frac{iz}{2} + C\bar{z}$$

This is what we wanted. Note that $\nabla \cdot \overline{H}$ and $\nabla \times \overline{H}$ are reals, so they are unchanged by conjugation.

p. 487, Note that if H is analytic, in which case $\overline{H}$ is sourceless and irrotational, then (17) correctly reduces to the first two terms of Taylor’s series: $H(z) = H(0) + H'(0)z + \dots$.

We are given that $\overline{C} = (1/2)[\partial_x H - i \partial_y H]$ and

$$\overline{H(z)} = \overline{H(0)} + (\nabla \cdot \overline{H}) \frac{z}{2} + (\nabla \times \overline{H}) \frac{iz}{2} + \overline{C}z \quad (17)$$

Since $\overline{H}$ is sourceless and irrotational, $\nabla \cdot \overline{H} = \nabla \times \overline{H} = 0$. Then

$$\overline{H(z)} = \overline{H(0)} + \overline{C}z$$

$$H(z) = H(0) + \overline{C}z \quad [\text{conjugate both sides}]$$

$$= H(0) + (1/2) [\partial_x H - i \partial_y H] z \quad [\text{replace } \overline{C}]$$

$$= H(0) + H'(0) z \quad [\text{by sum of (7) and (8), p. 210}]$$

Which is the first two terms of the Taylor's series

$$H(z) = H(0) + H'(0) z \dots$$

Pp. 487-488. Let K be a small simple loop of arbitrary shape round the origin, and let $\mathcal{A}$ be the area it encloses. We wish to show that the divergence and curl of H at 0 are the limiting values of the flux per unit area and work per unit area as K shrinks to the origin. Using (17) in (11) we find

$$W[\overline{H}, K] + i F[\overline{H}, K] = \oint_K H(z) dz \quad (11)$$

Before we can use (17) in (11), we must conjugate (17), which is an expression for $\overline{H(z)}$. Since flux density ($\nabla \cdot \overline{H}$) and curl ($\nabla \times \overline{H}$) are real numbers, they remain the same under conjugation.

$$H(z) = H(0) + (\nabla \cdot \overline{H}) \frac{z}{2} + (\nabla \times \overline{H}) \frac{iz}{2} + \overline{C}z$$

$$W[\overline{H}, K] + i F[\overline{H}, K] = \oint_K H(z) dz$$

$$= H(0) \int_K dz + \frac{1}{2} [\nabla \cdot \overline{H} - i \nabla \times \overline{H}] \int_K \bar{z} dz + \overline{C} \int_K z dz$$

$$= H(0) \cdot 0 + \frac{1}{2} [\nabla \cdot \overline{H} - i (\nabla \times \overline{H})] (2i \mathcal{A}) + \overline{C} \cdot 0$$

$$= [\nabla \times \overline{H} + i \nabla \cdot \overline{H}] \mathcal{A}$$

P. 488 Cauchy's formula

$H(z) = \frac{f(z)}{(z-p)}$. Note that $f(z)$ is analytic and $H(z)$ is analytic everywhere inside a simple loop K

except at p . This implies that $\overline{H}$ has vanishing flux and circulation densities **except at p .** See (12), and last paragraph, p. 483. *The Pólya vector field of H is divergence-free and curl-free if and only if H is analytic.* Needham provides the close approximation of Pólya vector field $\overline{H}$ very close to p .

$$\frac{A-iB}{\bar{z}-\bar{p}} = A\left[\frac{1}{\bar{z}-\bar{p}}\right] - B\left[\frac{i}{\bar{z}-\bar{p}}\right] \quad (1)$$

He presents a calculation of the Cauchy integral, where $f(p) = A + iB$.

$$\begin{aligned} \int_C \frac{f(z)}{(z-p)} dz &= W[\overline{H}, C] + iF[\overline{H}, C] \\ &= -2\pi B + i2\pi A \end{aligned} \quad (2)$$

The work W in (2) is $-2\pi B$ because, as explained, the vortex $i/(\bar{z} - \bar{p})$ has circulation of 2π "so this will also be its value for any simple loop round p " (p. 488). So we multiply $-B$ (the factor of the vortex) by 2π .

The flux F in (2) is $2\pi A$ because the first summand of the RHS of (1) "is familiar as a source at p of strength $2\pi A$..." (p. 488). I think we must take this on faith. It is a bit surprising that we can assign the value $F[\overline{H}, C] = 2\pi A$, because the flux density vanishes everywhere except at p , but Needham notes that a source is "pure flux", so it must be defined. It is only the flux density (divergence) that is not defined at p . Since p has no area, the flux density can not be defined (or it must be infinite).

Then we can rewrite (2).

$$\begin{aligned} &= 2\pi (-B + iA) \\ &= 2\pi i (A + iB) \\ &= 2\pi i f(p) \end{aligned}$$

P. 490, ¶ 3 *Let us now go beyond this geometric explanation in search of a compelling physical explanation. Figure [14b] shows the phase portrait of the dipole $(1/\bar{z}^2)$, the streamlines of which are apparently circular; a simple geometric argument [exercise] confirms their perfect circularity.*

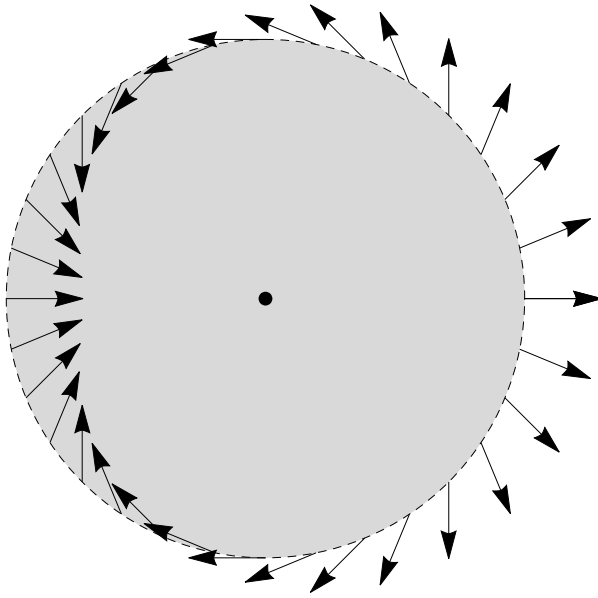
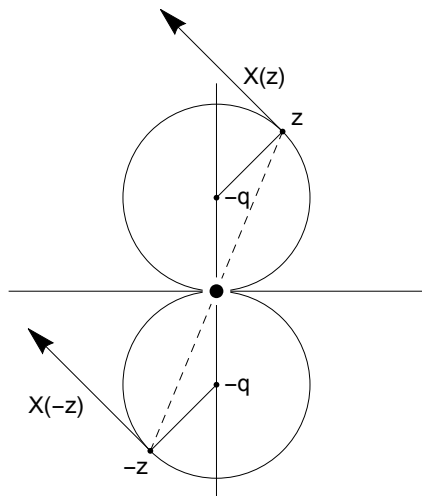
Figure 7. Vector field $1/(\bar{z}-\bar{p})^2$

Figure 7 shows the vector field $(1/\bar{z}^2)$ on a circle contour (because $p = 0$). If the field vector arrows along just the top semicircle are translated to the origin, they make a full rotation and their points describe a circle. Continuation of traversal around the circle produces another circle, but neither of the circles produced in this way are streamlines, because the vectors of X are tangent to their streamlines, and the circles are definitely not the streamlines of a dipole, because they are both centred on the origin and they are congruent.

Figure 8. Phase portrait of dipole for $1/(\bar{z}-\bar{p})^2$

That the streamlines of $(1/\bar{z}^2)$ are circles tangent to the real axis at the origin was established in Exercise 1 of Chapter 10, p. 468. It appears from the vector field that circles in both half planes are rotating in the same direction. This makes it difficult to understand how circles derived from the vector field can be the same as the circles of the dipole in [14b], p. 491, which shows the streamlines on opposite sides of the source/sink flowing in opposite directions. The answer seems to lie in how one finds the tangents to the streamlines, or rather how one interprets the vectors as tangents. The orientation of the streamline is determined by the direction of X on the dipole streamlines, not the direction of $X[z]$ on the contour or the direction of rotation of X on a contour traversed by z . Figure 2 shows X at z and $-z$. Given $X = (1/\bar{z}^2)$ and knowing that the streamlines are circles tangent to the real line at the origin, one can solve for the centres q and $-q$, which lie at the intersection of the y axis with $(i X(z))$ for q and with $(i X(-z))$ for $-q$. Then, the radius of the circles is just $|q|$ or $|z-q|$. Plot the circles. Since the tangents in the UHP and the LHP take opposite directions, we conclude that the streamlines have opposite directions of flow.

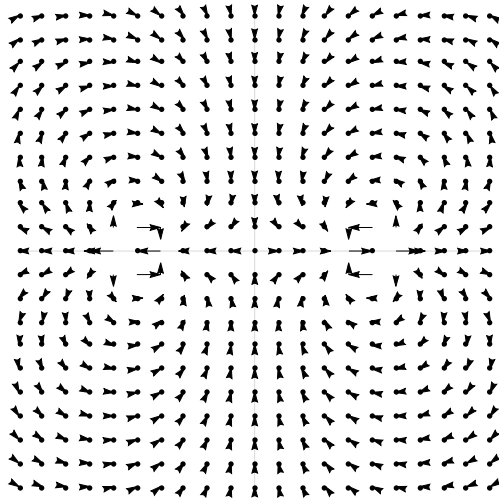


Figure 9. Phase portrait of $4d\epsilon\bar{z} / (\bar{z}^2 - \epsilon^2)^2$

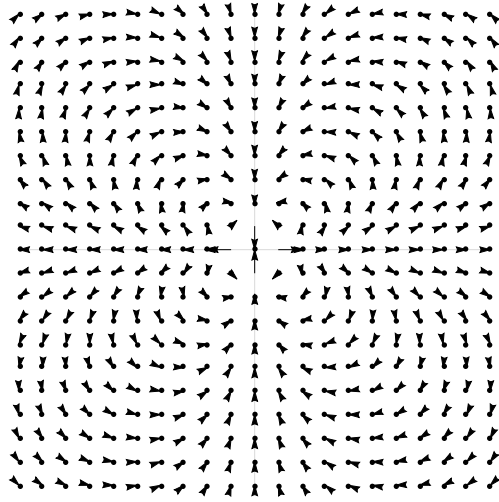
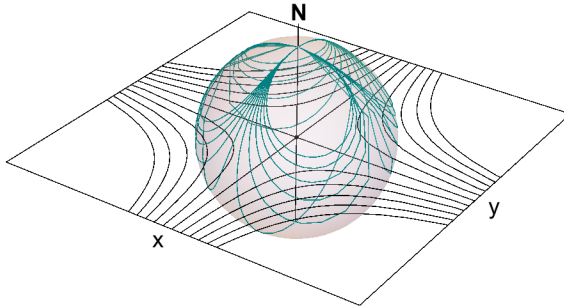
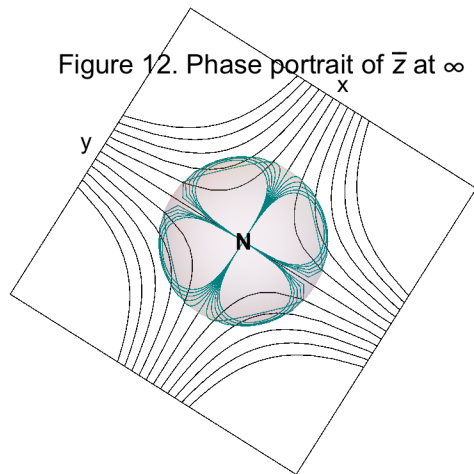


Figure 10. Phase portrait of quadrupole $k/\bar{z}^3$

P. 491-492, *The field for [15a] is $Q(z) = \dots = 4d\epsilon\bar{z}/(\bar{z}^2 - e^2)^2$. (19)*

Figure 9 verifies that $Q(z)$ in (19) plots the streamlines in [15a]. Figure 10 verifies that $k/\bar{z}^3$ plots as a quadrupole.

Figure 11. Phase portrait of $\bar{z}$ at ∞ Figure 12. Phase portrait of $\bar{z}$ at ∞

p. 493. *Given that z^0 yields a dipole at infinity, what might the Pólya vector field of z^l correspond to? Use your computer to see that it is a quadrupole at infinity.*

In Exercise 1 (ii), we learn that the streamlines of $\bar{z}$ near zero are hyperbolas with axes on the diagonals. $\bar{z}$ is the Pólya vector field of z^1 . In figure 11, we discover the shape of the streamlines under stereographic projection when ϵ goes to infinity. Actually, we were only able to send ϵ to a large number, but it was sufficient to illustrate the point. The unit sphere is centred at 0 in the complex plane. The hyperbola streamlines project to a quadrupole centred at N (Figures 11, 12).