

Notes on Needham, Chapter 11, Section I

G. Palmer, August 12, 2018, 5:30 pm PDT

Pp. 476–483. Much of this chapter apparently assumes the reader already knows the framework and ground rules of flow. The principles are introduced one by one, but interspersed with discussions and examples so smoothly that it is problematic to extract the general rules from the matrix of examples and the profusion of near synonyms. One could say that the chapter is example-driven rather than organized by the conceptual framework of flow.

The chapter characterizes flows described by analytic functions as sourceless and irrotational as opposed to flows with sources and rotation. A sourceless flow has zero flux density. An irrotational flow has zero work. These qualities are determined by measuring the flux density and work across a simple closed loop. Once a flow has been determined to be sourceless and irrotational, one can still calculate the integral of the flow H along a contour K that is not closed. And it is also the case that singularities may introduce sources and rotations into flows enclosed by simple loops.

The following is true for analytic functions.

Flow or field	math on simple loop	Physics on simple loop	Synonym
irrotational	zero curl	zero circulation, work	curl-free
sourceless	zero divergence	zero flux density, flux	divergence-free

On p. 484 the author considers the flow/field $\bar{z}$ that is not divergence-free. Perhaps we could say it has intrinsic divergence. With a simple calculation, the flow is shown to have flux density of 2. However, a contour K is required to calculate flux. There is an implied contour which we can liken to that on p. 477 or the contour enclosing an area on p. 484. It is also shown to be irrotational. On p. 488, Needham reintroduces Cauchy's formula, which has a pole (singularity in the denominator). He shows that for $f(z) = A - iB$, there is a source with strength $2\pi A$ and a vortex with strength 2π , and on p. 490 he states that *singularities are capable of generating flux and circulation*.

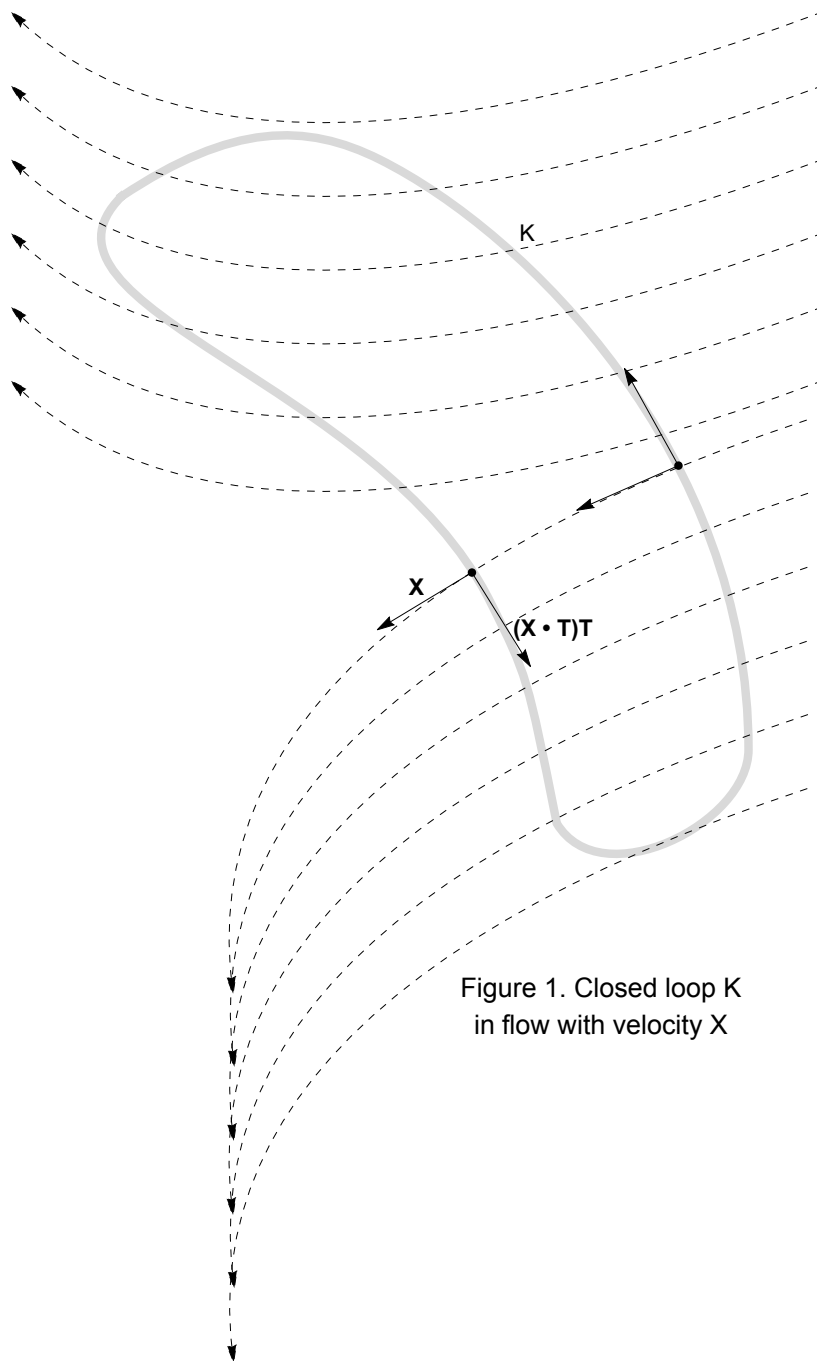


Figure 1. Closed loop K
in flow with velocity X

1. 2 Work, p. 476

Take K to be a closed loop and consider the thought experiment of Feynman[1963]. Imagine that the fluid flowing over the plane with velocity X is instantaneously frozen everywhere except within the narrow strip where our channel used to be. The “circulation” is then [exercise] the speed with which the unfrozen fluid flows (or circulates) round K , times the length of K .

The flow field is frozen everywhere except within the the strip where the channel once was. This strip is expanded from the shadow of the channel in [3b] to coincide with K itself (light gray). The resultant flow vector in K is the tangential component of X on K : $(X \cdot T) T$ (p. 474, line -6). The speed of flow at a point is the inner product $(X \cdot T)$.

In this situation, X is not “force”, and W is not “work”, but it is “circulation”, which seems to be a kind of “work” (p. 476, paragraph 2). As such, we should be able to calculate it with

$$W[X, K] = \int_K (X \cdot T) ds$$

which is the integral of the speed of flow over the loop. Then the speed of flow in the channel can be calculated as circulation over the distance

$$v_k = \int_K (X \cdot T) ds / (\text{length of } K)$$

and

$$W[X, K] = v_k (\text{length of } K)$$

We can apply this in an imprecise way to Figure 1. Since the initial flow is generally right to left by construction, the flow in K depends on the direction of T with signed magnitude $(X \cdot T)$. Though we can't be sure in this case, it appears that there is circulation throughout K .

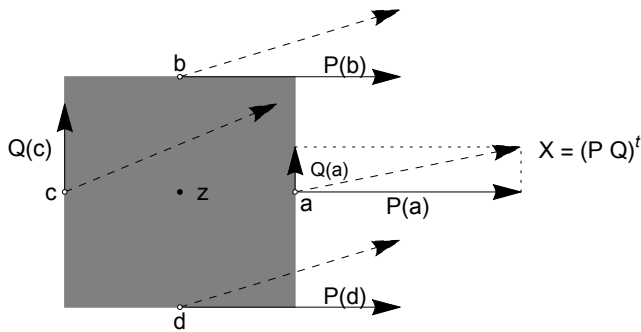


Figure 2. $W[X, \square] = \nabla \times X(z)$ (area of $\square$)

§§ 3 Local Flux and Local Work, p. 477. *Repeating the above analysis [of local flux] for the work, we find [exercise]*

$$W[X, \square] = [\nabla \times X(z)] \text{ (area of } \square) \quad (4)$$

where the formal cross product is defined by

$$\nabla \times X(z) = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \times \begin{pmatrix} P \\ Q \end{pmatrix} = \partial_x Q - \partial_y P.$$

We want the tangential components of X on $\square$. That is, we want the components of X that are parallel to the edges of the square. To understand the derivation, it is useful to note that P and Q are parallel to the axes as in Figure 2. That means that the sum $[P(b) - P(d)]$, where $b > d$, can be thought of as the canonical order of $\partial_y P(z)$. In the following derivation, the terms in P must be reversed to obtain this order, so we write $(-\partial_y P(z))$. This gives us the correct cross product. Referring to Figure [4] (Figure 2 above), we can write, proceeding anticlockwise around the square

$$\begin{aligned} W[X, \square] &= \epsilon Q(a) - \epsilon P(b) - \epsilon Q(c) + \epsilon P(d) \\ &= \epsilon [\{Q(a) - Q(c)\} - \{P(b) - P(d)\}] \\ &= \epsilon^2 [\partial_x Q(z) - \partial_y P(z)] \\ &= \epsilon^2 \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \times \begin{pmatrix} P(z) \\ Q(z) \end{pmatrix} \\ &= \nabla \times X(z) \text{ (area of } \square) \end{aligned}$$

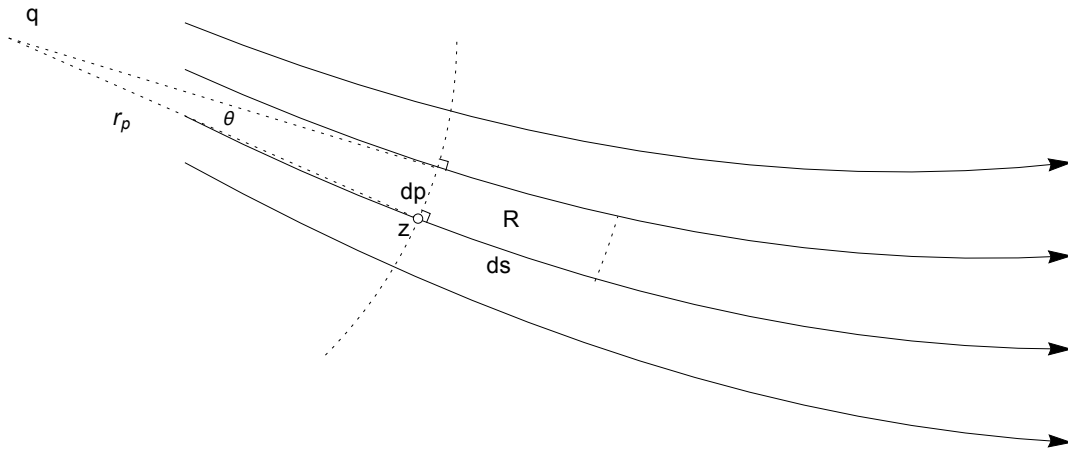


Figure 3. Flux and curvature

4 Divergence and curl in Geometric form, p. 478. If δ denotes the increase in a quantity as we move ds along the streamline, then [exercise] $\delta(dp) = k_p ds dp$.

From inspection of Figure 3, we can write:

$$\theta r_P = dp \implies \theta = dp / r_P$$

and

$$\delta(dp) = \theta (r_P + ds) - \theta r_P$$

$$= \theta ds$$

Substitute for θ .

$$\delta(dp) = (dp / r_P) ds$$

$$= \kappa_P dp ds \quad (\text{increase of } dp \text{ on } ds)$$

Thus, noting that (area of R) $\approx ds dp$, and $\delta\{\text{quantity}\}$ refers to the increase of a quantity on ds ,

$$(\text{Net flux out of R}) = \delta\{|X| dp\}$$

$$= (\delta|X|) dp + |X| \delta(dp) \quad (\text{Leibnitz})$$

$$= (\partial_s |X| ds) dp + |X| \kappa_P ds dp \quad (\text{step implicit in Needham, p. 479})$$

$$= (\partial_s |X| + \kappa_P |X|) (\text{area of R})$$

Since the net flux away from z is the flux density times the area of R, the flux density is

$$\nabla \cdot X = (\partial_s |X| + \kappa_P |X|)$$

p. 479. *In fact [exercise] this formula is still true for a three dimensional vector field, provided that P is taken to be the surface orthogonal to the streamlines, and κ_P is taken to be the sum of its principle curvatures.*

Let dp be the change in surface area p with motion along s . Let C be a cube with one surface equal to dp and orthogonal edge s passing through z parallel to the streamlines of X . Let the two edges of dp be dp_1 and dp_2 , so that $dp = dp_1 dp_2$. Then

$$\delta(dp) = \delta(dp_1 dp_2) = \delta(dp_1) dp_2 + dp_1 \delta(dp_2) \quad (\text{Leibnitz})$$

$$= (\kappa_{P1} ds dp_1) dp_2 + dp_1 (\kappa_{P2} ds dp_2) \quad (\text{from previous exercise})$$

$$= (\kappa_{P1} + \kappa_{P2}) dp ds$$

$$= k_P \, dp \, ds$$

$$(\text{Net flux out of } C) = \delta \{ |X| \, dp \}$$

$$= (\delta |X|) \, dp + |X| \, \delta(dp) \quad (\text{Leibnitz})$$

$$= (\partial_s |X| \, ds) \, dp + |X| \, k_P \, dp \, ds$$

$$= (\partial_s |X| + k_P |X|) (\text{volume of } C)$$

$$= \nabla \cdot X \, (\text{volume of } C)$$

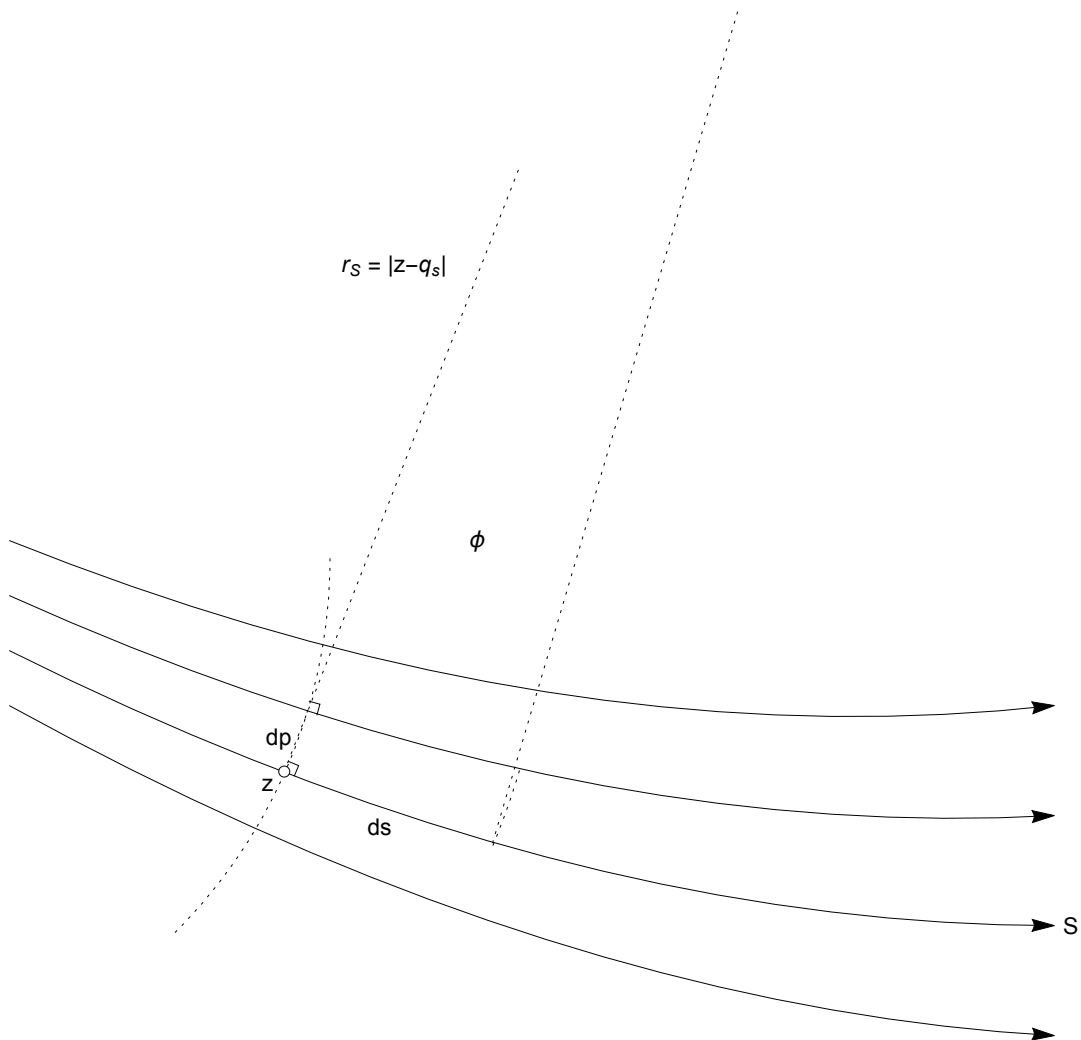
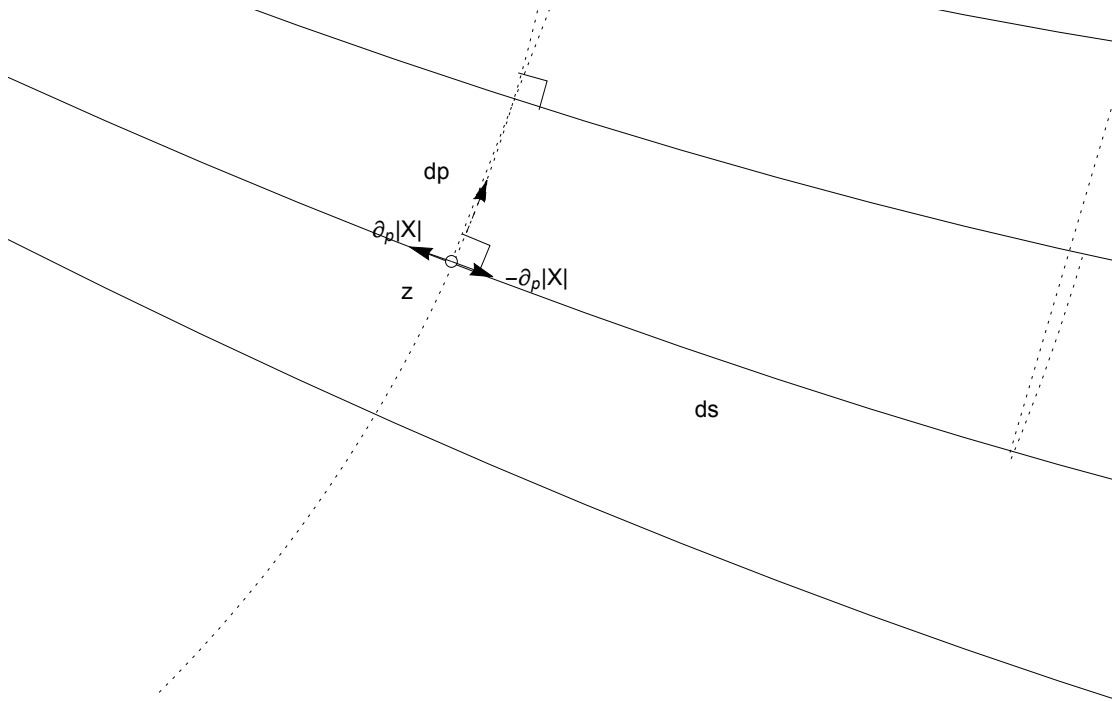


Figure 4a. Curl and curvature

Figure 4b. Partial of $|X|$ on p , orthogonal to S

P. 479, Turning to the circulation round R , identical reasoning yields [exercise] an equally neat formula for the curl:

$$\nabla \times X = -\partial_p |X| + \kappa_S |X|, \quad (6)$$

where κ_S is the curvature at z of the streamline S .

Assumptions: dp and ds are absolute values (like r) representing length of arc or curve. They have no inherent orientation (unlike θ and ϕ). Thus $(r_S - dp)$ is a shorter distance than r_S . $\delta(ds)$ refers to increase in ds .

We will need an expression for $\delta(ds)$. Let r_S be the radius of curvature of S at z (Figure 4a).

$$ds = \phi r_S \quad \Rightarrow \quad \phi = ds / r_S$$

$$\delta(ds) = \phi r_S - \phi (r_S - dp) \quad (\text{increase in } ds \text{ along } dp, N \text{ to } S)$$

$$= \phi dp$$

$$= (ds / r_S) dp = \kappa_S ds dp$$

The circulation involves the sum of the circulation (speed times arc length) on all four sides of R ,

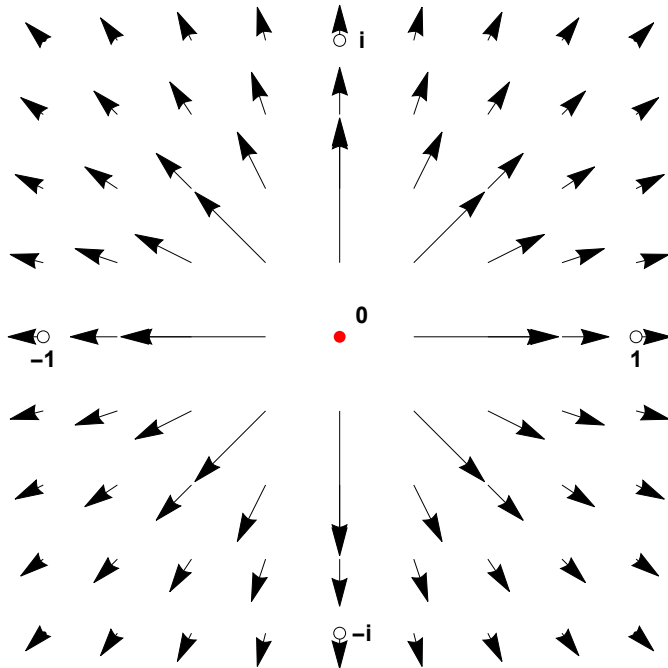
which we designate by the labels N, S, E, and W, where $W = dp$ and $S = ds$. Since the field vectors all point in the direction of the streamlines, which are orthogonal to the arc dp , there is no circulation on sides E or W. The net circulation (work) is the sum $W[X, N] + W[X, S]$ (this W for work not to be confused with W for dp above). Circulation is affected by two factors: (1) the increasing speed of X along the orthogonal dp with movement from north to south, that is, from $(z + \theta r_p)$ to z , and (2) the increasing speed of X along ds with increasing r_p . For illustrations of the increase in speed on a streamline, see Figures [5] and [8]. We can see from [5] that ds is longer than the streamline above it, which has negative curl, so net curl in the anticlockwise direction is likely to be positive.

The net increase in circulation on R is therefore

$$\begin{aligned}
 (\text{Net work/circulation about R}) &= \delta(|X| \, ds) && (\text{increase in speed of } |X| \text{ over arc } ds) \\
 &\quad (1) \qquad (2) && (\text{the two factors}) \\
 &= \delta(|X|) \, ds + |X| \, \delta(ds) && (\text{Leibnitz!}) \\
 &= [-\partial_p |X| \, dp] \, ds + |X| \, (\kappa_S \, ds \, dp) && (\text{Compare Figure 4b}) \\
 &= (-\partial_p |X| + \kappa_S |X|) (\text{area of R}) && (\text{area} = dp \, ds)
 \end{aligned}$$

Since the net circulation/work around R is the curl times the area of R, the curl is

$$\nabla \times X = (-\partial_p |X| + \kappa_S |X|)$$

Figure 5. $X(z) = S/(2\pi i \bar{z})$

P. 479, §§5.

$$X(z) = \frac{S}{2\pi \bar{z}} \iff X = \frac{S}{2\pi} \begin{pmatrix} x/(x^2 + y^2) \\ y/(x^2 + y^2) \end{pmatrix}$$

This should have zero flux density (i.e. divergence) everywhere except at the origin, where it should be undefined. Check that this is so.

See Exercise 1 (i) for an example of the derivation of P and Q for $X = \frac{1}{\bar{z}}$. Then multiply the result by $\frac{S}{2\pi}$.

We can check by plotting the field on a grid (Figure 5). Anytime $z = 0$, $\bar{z}$ and $(x^2 + y^2)$ are also zero, so X is undefined at 0, as are flux and curl. Let $r = (x^2 + y^2)$

Calculation of local flux density and net flux:

$$X = \frac{S}{2\pi} \begin{pmatrix} x/(x^2 + y^2) \\ y/(x^2 + y^2) \end{pmatrix}$$

Then $P = \frac{S}{2\pi} (x/(x^2 + y^2))$ and $Q = \frac{S}{2\pi} (y/(x^2 + y^2))$.

Local flux density $[\nabla \cdot \mathbf{X}(z)] = [\partial_x P(z) + \partial_y Q(z)] = 0$

$$\begin{aligned}
 & \partial_x \left(\frac{S}{2\pi} (x / (x^2 + y^2)) \right) + \partial_y \left(\frac{S}{2\pi} (y / (x^2 + y^2)) \right) \\
 &= \frac{S}{2\pi} \partial_x [x (x^2 + y^2)^{-1}] + \frac{S}{2\pi} \partial_y [y (x^2 + y^2)^{-1}] \quad \text{use Leibnitz} \\
 &= \frac{S}{2\pi} [1 (x^2 + y^2)^{-1} + x (- (x^2 + y^2)^{-2} 2x] + \frac{S}{2\pi} [1 (x^2 + y^2)^{-1} + y (- (x^2 + y^2)^{-2} 2y)] \\
 &= \frac{S}{2\pi} \left(\frac{1}{r^2} - 2 x^2 \frac{1}{r^4} + \frac{1}{r^2} - 2 y^2 \frac{1}{r^4} \right) \quad \text{replace} \\
 & \quad (x^2 + y^2)^{1/2} \text{ with } r \\
 &= \frac{S}{2\pi} \left(\frac{2}{r^2} - 2 r^2 \frac{1}{r^4} \right) = 0
 \end{aligned}$$

Calculation of local curl and work:

Local curl $[\nabla \times \mathbf{X}(z)] = [\partial_x Q(z) - \partial_y P(z)] = 0$

$$\begin{aligned}
 & \partial_x \left[\frac{S}{2\pi} y (x^2 + y^2)^{-1} \right] - \partial_y \left[\frac{S}{2\pi} x (x^2 + y^2)^{-1} \right] \\
 &= \frac{S}{2\pi} [0 - y (x^2 + y^2)^{-2} 2x] - \frac{S}{2\pi} [0 - x (x^2 + y^2)^{-2} 2y] \\
 &= \frac{S}{2\pi} [-xy (x^2 + y^2)^{-2} + xy (x^2 + y^2)^{-2}] = 0
 \end{aligned}$$

P. 480, line -1. *What does the second one say?*

The total amount of fluid curling around K is the sum of the flow curling around the interior squares, where adjacent sides of interior squares cancel.