

Needham, Chapter 11, Exercise 9, p.

G. Palmer, September 12, 2018, 11:40 pm PDT

(i) Explain why the derivative of the complex potential of a source yields the complex potential of a dipole.

The complex potential is the Pólya work plus flux, which is the same as the integral of the vector field on a contour. Then, one might use $\overline{H} = A/(\bar{z}-\bar{p})$ for a source (from p. 488), where A is a real number, as one can deduce from its role in Cauchy's formula (p. 489). Let $p = 0$, which gives a source vector field (not Pólya vector field) of $H = A/z$. We will need this form to describe the integral. A negative value for A would correspond to a sink.

On p. 491, we find a general equation for a dipole,

$$D(z) = \frac{ke^{-i\phi}}{\bar{z}^2}$$

“i.e. the general dipole field obtained by rotating [14] by $+\phi$ and scaling up the speed of the flow by k, which we may think of as the ‘strength’ of the dipole. Thus, the Pólya vector field of (d/z^2) is a dipole whose axis points in the direction of d, and whose strength is |d|. The complex number d is called the *dipole moment*.” From this we learn that a dipole such as D(z) is regarded as a Pólya vector field and that the *dipole moment* pertains to the vector field itself, not the Pólya vector field. If $\overline{d/z^2} = ke^{-i\phi}/\bar{z}^2$, then $\overline{d} = ke^{-i\phi}$ and $d = ke^{i\phi}$. For a specific example, on p. 503 (1), we find a dipole described as $\overline{H} = \frac{1}{\bar{z}^2}$ (the z is conjugate).

This $\overline{H}$ notation for the dipole seems to justify using $\overline{H}$ for the Polya field of the source as well as the dipole, as we did above. See also, (3), p. 503. Since the source has a vector field $H = A/z$, we can apply (11).

$$\int_K H(z) dz = W[\overline{H}, K] + i F[\overline{H}, K] = \Omega(z)$$

$$\Omega_S(z) = \int_K (A/z) dz \text{ [p. 503, (3)], where } \Omega_S \text{ pertains to a source}$$

with derivative $(\int_K (A/z) dz)' = A/z$, treating A as a constant,

$$\Omega'_S(z) = A/z + \text{const.}$$

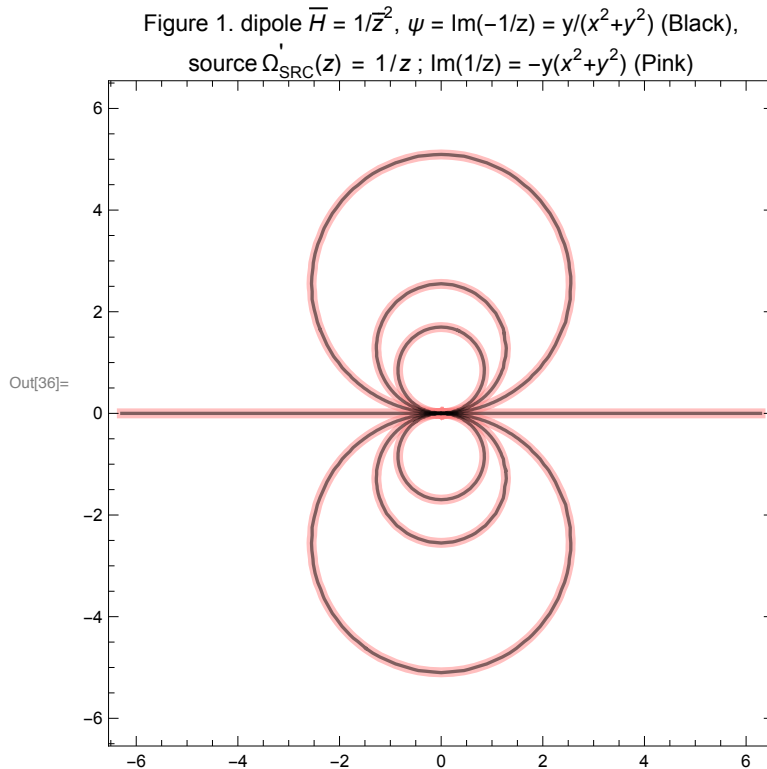
For a dipole $\overline{H} = 1/\bar{z}^2$, with vector field d/z^2 the complex potential is

$$\Omega_{DP}(z) = \int_K (d/z^2) dz, \text{ treating } d \text{ as a (complex) constant}$$

$$= -d/z + \text{const.}$$

For example, we note that on p. 503 (1), $\Omega_{DP} = -1/z + c$ [p. 503, (1)]. We use this complex potential for the dipole in Figure 1.

Thus, $\Omega'_S(z) = \Omega_{DP}(z)$ if $A = -d$. To be precise, $\Omega'_S(z)$ has the same format as a dipole for which d is real because $\phi = 0$. The streamlines of a dipole and the derivative of the complex potential of a source are plotted in Figure 1. The dipole is plotted in black and transposed onto the thick pink background of the derivative of the complex potential of the source. Even though they have different formulae, Figure 1 shows that their shapes are the same.



(ii) Referring to the previous question, draw a sketch predicting the appearance of the flow whose complex potential is $\Omega(z) = \frac{d}{dz} \ln[\sin z]$. Check your answer by getting the computer to draw this flow.

From part (i) and from Exercise 8, one might guess that $\Omega = \frac{d}{dz} \ln[\sin z]$ is the derivative of the complex potential of the sources $\pm n\pi$, $n = 1.. \infty$. Then one could sketch a dipole at every source.

$$\Omega(z) = \frac{d}{dz} \ln[\sin z] = \int_a^z H dz = \phi(z) + i \psi(z)$$

$$= \cot(z)$$

Take real and imaginary parts,

$$\phi = \operatorname{Re}(\cot(z))$$

$$\psi = \operatorname{Im}(\cot(z))$$

The plot of the streamlines ψ of $\Omega(z)$ reveals that the poles of the dipoles act as sources and sinks for their nearest neighbors as well as for their own flows. As sinks, they seem to greedily suck in all the available flow from their nearest neighbors. None of it skips over to more distant sinks.

Figure 2. Streamlines of $\psi = \operatorname{Im}(\cot(z))$

$$\psi = -2\pi \dots 2\pi \text{ by } \pi/16$$

