

Needham, Chapter 11, Exercise 8, p. 506.
G. Palmer, September 11, 2018, 12:30 pm PDT

8 Consider the flow created by $(2n + 1)$ sources, each of strength 2π , located at

$$0, \pm\pi, \pm2\pi, \dots, \pm n\pi.$$

(i) If $\Omega_n(z)$ denotes the complex potential of this flow, show that

$$\Omega_n(z) = \ln \left[z \left(1 - \frac{z^2}{\pi^2} \right) \left(1 - \frac{z^2}{2^2 \pi^2} \right) \dots \left(1 - \frac{z^2}{n^2 \pi^2} \right) \right] + \text{const.}$$

Since Ω_n for this problem is a complex log, we will rewrite it as “log”. What is meant by “strength 2π ”? On p. 453 in Subsection 4 Sources and Sinks, we find “Thinking in terms of our layer of ideal fluid, a source of strength S is a point at which we pump in S units of fluid per unit of time.” On p. 455, S is defined as

$$S = 2\pi|V|$$

where V is “the component of the velocity perpendicular to the curve.”

$$V(z) = |V| e^{i\theta} = \frac{S}{2\pi} \left(\frac{e^{i\theta}}{r} \right) = \frac{S}{2\pi} \left(\frac{1}{z} \right)$$

In the current problem, no curve is mentioned. Evidently, the singularities in this problem are all sources, each with a velocity

$$V(z) = \frac{S}{2\pi} \left(\frac{1}{z} \right) = \frac{2\pi}{2\pi} \left(\frac{1}{z} \right) = \frac{1}{z}$$

What is velocity? on P. 472, it is “a vector field X (which we will first think of as the velocity of a fluid flowing over the plane)”. On p. 481, the notation “ H ” was introduced for a vector field in connection to the Pólya vector field $\overline{H}$. So it appears that $V(z)$, $X(z)$, and $H(z)$ are equivalent, at least for some purposes. We know that $H = \Omega'$ (p. 502). We note also that part (iii) of this problem has $V = \overline{\Omega'}$. Then $\overline{V}(z) = \Omega'$ and $\Omega = \int_K \overline{V}(z) dz$ for some contour K . Since this problem stipulates sources at $\pm n\pi$, we can write

$$V_n(z) = \frac{1}{z - (\pm n\pi)} = \frac{1}{z \mp n\pi}$$

In the present case, we have $(2n + 1)$ sources and no sinks. Perhaps we can merely add them together. Let n be the number of $\pm$ pairs of source locations.

$$\begin{aligned} \mathcal{V}_n(z) &= \sum_1^n V = V_0(z) + \sum_1^n (V_{-n} + V_n) \\ &= \frac{1}{z} + \sum_1^n \left(\frac{1}{z - n\pi} + \frac{1}{z + n\pi} \right) \\ &= \frac{1}{z} + \sum_1^n \frac{(\overline{z} + n\pi) + (\overline{z} - n\pi)}{(\overline{z} - n\pi)(\overline{z} + n\pi)} \end{aligned}$$

$$= \frac{1}{z} + \sum_1^n \frac{2\bar{z}}{z^2 - n^2 \pi^2}$$

Then the Pólya vector field of the velocity vector field is

$$\overline{\mathcal{V}_n(z)} = \frac{1}{z} + \sum_1^n \frac{2z}{z^2 - n^2 \pi^2}$$

We will assume that $\Omega'_n = \overline{\mathcal{V}_n(z)}$.

$$\begin{aligned} \Omega'_n &= \frac{1}{z} + \sum_1^n \frac{2z}{z^2 - n^2 \pi^2} \\ &= \frac{1}{z} + \sum_1^n \frac{d}{dz} \log(z^2 - n^2 \pi^2) \end{aligned}$$

Integrate along a path from 0 to z that avoids integers, renaming the variable from z to w to permit integration on a range including z and obtain a result in z .

$$\int_0^z \Omega'_n(w) dw = \int_0^z \left[\frac{1}{w} + \sum_1^n \frac{d}{dw} \log(w^2 - n^2 \pi^2) \right]$$

$$\Rightarrow \Omega_n(z) = \log(z) + \sum_1^n \log(z^2 - n^2 \pi^2) + \text{const.}$$

$$= (\log[z (z^2 - \pi^2)(z^2 - 2^2 \pi^2)(z^2 - 3^2 \pi^2) \dots] + \text{const.})$$

Divide each factor other than z by $-n^2 \pi^2$ and multiply the result by $(-n^2 \pi^2)^n$.

$$\begin{aligned} \Omega_n(z) &= \log\left[(-n^2 \pi^2)^n z \left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{2^2 \pi^2}\right) \dots \left(1 - \frac{z^2}{n^2 \pi^2}\right)\right] + \text{const.} \\ &= \log\left[z \left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{2^2 \pi^2}\right) \dots \left(1 - \frac{z^2}{n^2 \pi^2}\right)\right] + n \log(-n^2 \pi^2) + \text{const.} \\ &= \log\left[z \left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{2^2 \pi^2}\right) \dots \left(1 - \frac{z^2}{n^2 \pi^2}\right)\right] + \text{const.} \end{aligned}$$

This is what we wanted.

(ii) Ignoring the constant, and referring to Ex. 13 on p. 449, deduce that as the number of sources increases without limit, $\Omega_n(z)$ tends to $\Omega(z) = \ln[\sin z]$.

On p. 449, the notation of (iv), ignoring the constant, indicates the $\ln$ of a product of z times an infinite number of $(1 - \frac{z^2}{n^2 \pi^2})$ terms. These are the same terms that we obtain in (i) of this problem if n is allowed to go to infinity. Since the product equals $\sin z$, and n is the number of pairs of sources, $\Omega(z) = \log(\sin z)$.

(iii) Check that this answer makes sense by using a computer to draw the velocity vector field $V = \overline{\Omega'}$.

$$(\log(\sin z))' = \cot z$$

$$\overline{\Omega'} = \overline{\cot z}$$

The answer, shown in Figure 1, appears to make sense because arrows point away from the sources specified in the problem. Vasco has plotted the gridpoints so as to divide the distance between two sources by 8. This choice reveals that the arrows emanating from the midpoints between sources are vertically aligned, pointing either up or down (Figure 2). He concluded that the flow is confined to channels of width π . To illustrate, he plotted the lines of flow. We can do that by dividing $\Omega = \log(\sin z)$ into real and imaginary parts. Then $\psi = \text{Im}(\Omega)$ is plotted with a contour plot.

$$\Omega(x, y) = \ln|\sin(z)| + i \arg(\sin(z))$$

$$\psi = \arg(\sin(z))$$

Mathematica plots the contours, but it doesn't do a great job (Figure 3). One can produce an accurate looking plot, using $\psi = \arg(\sin(z))$, but it has odd gaps (Figure 3). With $\psi = \arctan(\text{Im}(\sin(z))/\text{Re}(\sin(z)))$, the gaps disappear. An apparently identical plot can be obtained with $\psi = \arctan(\cos x \sinh y/(\sinh x \cosh y))$

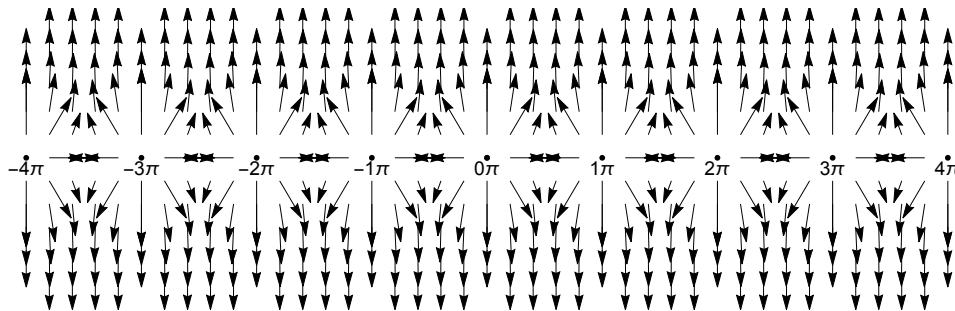


Figure 2. $\overline{\Omega'} = \frac{d}{dz} \log(\sin z) = \overline{\cot z}$

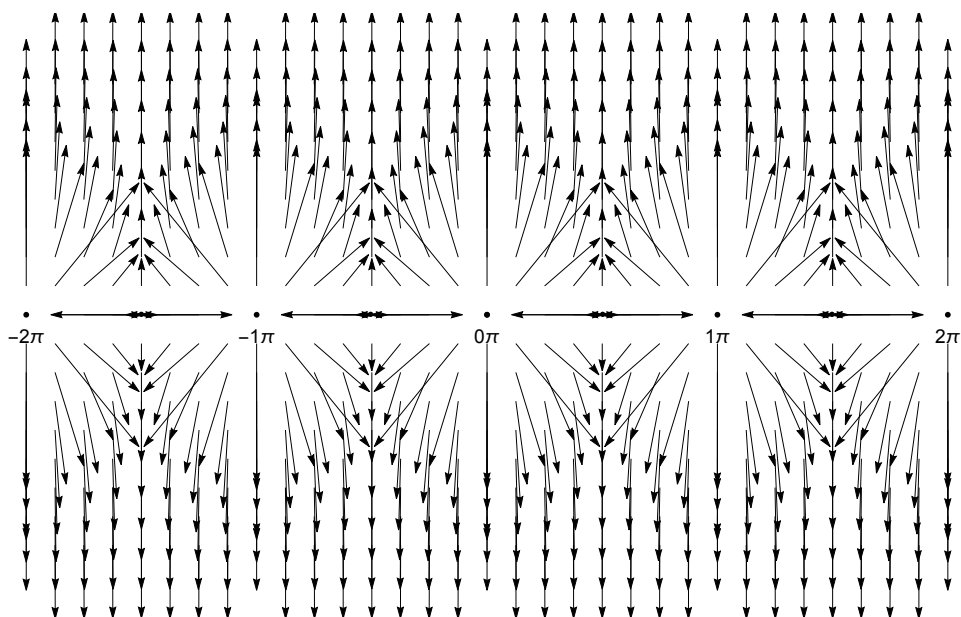


Figure 2. $\overline{\Omega}' = \overline{\frac{d}{dz} \log(\sin z)} = \overline{\cot z}$

Figure 3. $\psi = \arg(\sin(z))$ 