

Needham, Chapter 18, Exercise 7, p. 506.
 G. Palmer, Nov 21, 2018, 11 am PST

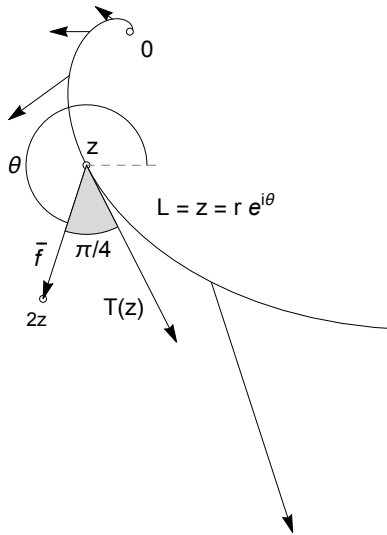


Figure 1. Explanation of
 Braden's Theorem
 $f(z) = \bar{z}$, $r = e^\theta$

7 Continuing from the previous question, suppose that $f(z) = \bar{z}$.

(i) Show that if L is a segment of the spiral with polar equation $r = e^\theta$, then the condition of Braden's Theorem is met.

$$\begin{aligned} f &= \bar{z} \\ \bar{f} &= z \\ r &= e^\theta \\ z &= r e^{i\theta} = e^\theta e^{i\theta} = e^{(1+i)\theta} \\ L &= z(\theta) = r(\theta) e^{i\theta} \end{aligned}$$

L is the contour traversed by z , so z and L must have the same equations. z is also the Pólya vector field $\bar{f}$ of $f(z) = \bar{z}$, so we plot z on L , which is also plotting $z(\theta)$ on $z(\theta)$. $\text{Arg}(\bar{f}) = \theta$ by the definition of z . The angle of the contour at z is $\text{arg}(T(z))$. Then we want to show that $(\text{arg}(T) - \text{arg}(\bar{f}))$ is constant on L . Since $\text{arg}(T) = \text{arg}(dz/d\theta)$, find the tangent vector $dz/d\theta$.

$$\begin{aligned} \frac{dz}{d\theta} &= (1+i) e^{(1+i)\theta} \\ &= (1+i) e^\theta e^{i\theta} = e^\theta (1+i) e^{i\theta} \end{aligned}$$

$$\begin{aligned}
\arg(T) - \arg(\bar{f}) &= \arg(T) - \theta \\
&= \arg(e^\theta(1+i) e^{i\theta}) - \theta \\
&= \arg(1+i) + \arg(e^{i\theta}) - \theta \quad [\arg(e^\theta) = 0] \\
&= \pi/4 + \theta - \theta = \pi/4
\end{aligned}$$

This result meets the conditions of Braden's Theorem.

(ii) Verify by explicit calculation that equality does indeed hold in (32), as predicted.

$$\begin{aligned}
z &= re^{i\theta} = e^\theta e^{i\theta} \\
dz &= (1+i) e^{(1+i)\theta} d\theta
\end{aligned}$$

Let α and β be initial and final angles of z traversing L .

$$\begin{aligned}
\left| \int_L f(z) dz \right| &= \left| \int_L \bar{z} dz \right| = \left| \int_\alpha^\beta (e^\theta e^{-i\theta}) [(1+i) e^{(1+i)\theta} d\theta] \right| \\
&= \left| \int_\alpha^\beta (1+i) e^{2\theta} d\theta \right| \\
&= \left| \frac{(1+i)}{2} [e^{2\theta}]_\alpha^\beta \right| \\
&= \frac{\sqrt{2}}{2} |e^{2\beta} - e^{2\alpha}|
\end{aligned}$$

$$\begin{aligned}
\int_L |f(z)| |dz| &= \int_L |\bar{z}| |dz| = \int_\alpha^\beta |e^\theta e^{-i\theta}| |(1+i) e^{(1+i)\theta} d\theta| \\
&= \int_\alpha^\beta e^\theta |(1+i) e^{i\theta} d\theta| \\
&= |(1+i)| \int_\alpha^\beta e^{2\theta} d\theta \\
&= \frac{|(1+i)|}{2} (e^{2\beta} - e^{2\alpha}) \\
&= \frac{\sqrt{2}}{2} |e^{2\beta} - e^{2\alpha}|
\end{aligned}$$

Since $|d\theta|$ is positive, β is a larger angle than α , so $(e^{2\beta} - e^{2\alpha}) = |e^{2\beta} - e^{2\alpha}|$. The equality in (32) is verified by explicit calculation.