

Needham, Chapter 11, Exercise 6, p. 505-506.

G. Palmer, November 15, 2018, 2:30 pm PST

6 All complex analysis texts recognize the great utility of the inequality

$$|\int_L f(z) dz| \leq \int_L |f(z)| \cdot |dz| \quad (32)$$

but none that we know of have sought to answer the question “When does equality hold?” This is probably because no elegant answer is forthcoming (cf. our attempt in Chapter 7) without the concept of the Pólya vector field. However, armed with the Pólya vector field, we have what we shall call Braden’s Theorem....:

“Equality holds in (32) if and only if the contour L cuts the streamlines of the Pólya vector field of f at a constant angle.”

Explain Braden’s Theorem

A Pólya vector field may or may not have a stream function, and so it may or may not have streamlines. Vasco has amended the theorem to read:

Equality holds in (32) if and only if the contour L makes a constant angle with the Pólya vector field of f .

This proof pertains to the corrected theorem.

Figure [7], p. 482 appears to provide the best framework for this problem. We let $H = f(z)$, which is a field vector along the contour $K = L$, which may or may not be closed. dz is a vector tangent to L . The contour L cuts $\bar{f}$ at the angle θ , the angle between $\bar{f}$ and the tangent to L for any z . $\bar{f}$ is useful for this explanation, because $\int_L f(z) dz$ can be written in terms of dot products of $\bar{f}$ with T and N , which depend on L (p. 482).

$$|\int_L f(z) dz|$$

On p. 482, we find that $H dz = [\bar{H} \cdot T + i \bar{H} \cdot N] ds$, where $H = [\bar{H} \cdot T + i \bar{H} \cdot N]$ and ds is the length of a small segment of L . At very small values of ds , $ds = |dz|$. $T(z)$ and $N(z)$ are unit vectors with angles that depend only on L at z . We will use $a \cdot b = |a||b| \cos(\theta)$.

$$\begin{aligned} |\int_L f(z) dz| &= |W[\bar{H}, K] + i F[\bar{H}, K]| \\ &= |\int_L [(\bar{f}(z) \cdot T) + i \int_K (\bar{f}(z) \cdot N)] |dz|| \\ &= |\int_L [|\bar{f}(z)| \cos(\theta) + i \int_K |\bar{f}(z)| \cos(\theta - \pi/2)] |dz|| \end{aligned}$$

$$\begin{aligned}
&= \left| \int_L [\overline{f}(z) (\cos(\theta) + i \sin(\theta))] dz \right| \\
&= \left| \int_L \overline{f}(z) e^{i\theta} dz \right| \\
&= \left| \int_L e^{i\theta} \overline{f}(z) dz \right| \leq \left| e^{i\theta} \int_L \overline{f}(z) dz \right| \quad (1)
\end{aligned}$$

Explanation: $\int_L \overline{f}(z) dz$ is composed of a large number of summands with angle θ , which may be different for each, or it may be constant. Adding vectors with varying angles end to end as required by the Riemann integral produces an integral $\left| \int_L e^{i\theta} \overline{f}(z) dz \right| \leq \left| e^{i\theta} \int_L \overline{f}(z) dz \right|$ (See [9], p. 385). If θ is constant, as on the RHS, the angles of the summands are equal and the vectors align to form a straight line segment, which is the longest possible by the triangle inequality, in which case, $\left| \int_L f(z) dz \right| = \left| e^{i\theta} \int_L \overline{f}(z) dz \right|$.

$$\int_K |f(z)| \cdot |dz|$$

While there appears to be an inner product sign in the integral on the RHS of (32), the result is equivalent to the product of real numbers $\int_K (|f(z)| \cdot |dz|)$, so the notation needn't concern us. Both $|f(z)|$ and $|dz|$ are real numbers, so both have angle 0. The integral of these real numbers is a real number which equals the absolute value of the integral with the constant angle (1). Since $|f(z)||dz|$ is the same for the LHS and the RHS of (2), and since the real values of the RHS are all aligned and added at the same angle, and since $|e^{i\theta}| = 1$, we can write

$$\int_K (|f(z)| \cdot |dz|) = \left| e^{i\theta} \int_L \overline{f}(z) dz \right| \quad (\text{The maximum integral.}) \quad (2)$$

We have shown that $\left| \int_L f(z) dz \right| \leq \left| e^{i\theta} \int_L \overline{f}(z) dz \right|$, where equality holds only if the contour L the Pólya vector field of $f(z)$ at a constant angle. We have also shown that $\left| e^{i\theta} \int_L \overline{f}(z) dz \right| = \int_K (|f(z)| \cdot |dz|)$. It follows that $\left| \int_K f(z) dz \right| \leq \int_K |f(z)| \cdot |dz|$ if and only if the contour L cuts the Pólya vector field of f at a constant angle. This proves Braden's Theorem as amended.