

Needham, Chapter 18, Exercise 5, p. 505.

G. Palmer, May 21, 2018, 10:15 am PDT

5 Let L be a contour from the real number $-\theta$ to θ . By choosing L to be a line-segment, and then sketching the Pólya vector field at points along L , show that $\int_L z e^{iz} dz$ is purely imaginary. Verify this by calculating the exact value of the integral.

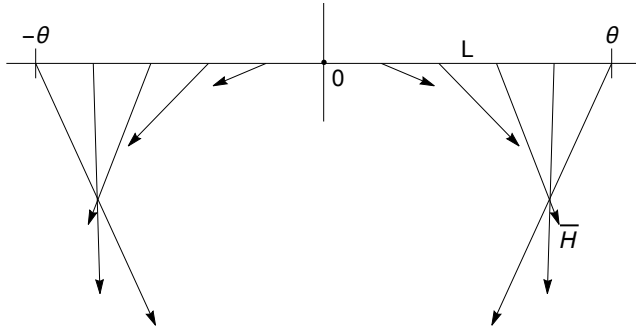


Figure 1. Pólya vector field on $-1 \leq \theta \leq 1$

Figure 1 sketches the Pólya vectors on the line segment $-1 \leq \theta \leq 1$. Figure 1 shows that the Pólya vectors (black) are symmetrical about the y axis. dz is a constant real number. Hence, $\text{Re}(\sum \overline{H}) dz = 0$. Since the H vectors are conjugates of the Pólya vectors, we can also deduce that $\text{Re}(\sum H) dz = 0$. Adding the $(H dz)$ vectors, we get $\sum H dz = 1.0532 i$. Using the fundamental theorem of calculus, we calculate $\int_L z e^{iz} dz = 0.602337 i$. Thus, the integral of the vector field is entirely on the imaginary axis. There is only flux.

Calculation of $\int_L z e^{iz} dz$

Let $z = x + iy$. Since θ is on the real line, and the Pólya vector field is sketched at points on the line segment $-\theta \dots \theta$, we integrate on x from $-\theta$ to θ . Since $y = 0$ on the real line, $z = x$ and $dz = dx$. Let, $u = x$, $v = e^{ix}$. Then $du = dx$, $dv = i e^{ix} dx$, $dv = x$.

$$\int u dv = uv - \int v du$$

$$\int_L z e^{iz} dz = \int_{-\theta}^{\theta} x e^{ix} dx = -i \int_{-\theta}^{\theta} x i e^{ix} dx$$

$$= -i [x e^{ix}]_{-\theta}^{\theta} - \int_{-\theta}^{\theta} e^{ix} dx$$

$$= -i [\theta e^{i\theta} - (-\theta e^{i(-\theta)}) + i e^{ix} \big|_{-\theta}^{\theta}]$$

$$= -i [\theta(e^{i\theta} + e^{-i\theta}) + i e^{i\theta} - i e^{i(-\theta)}]$$

$$= -i[2 \theta \cos \theta + i (2i) \sin \theta]$$

$$= -2i\theta \cos \theta + 2 i \sin \theta$$

$$= -2i(\theta \cos \theta - \sin \theta)$$

Since θ , $\cos \theta$, and $\sin \theta$ are all real, the integral must always be purely imaginary.

When $\theta = 1$, $\int_L z e^{iz} dz = 0.602337 i$.

When $\theta = \pi$, $\int_L z e^{iz} dz = 2 \pi i$.

When $\theta = 2\pi$, $\int_L z e^{iz} dz = 4 \pi i$.

It seems odd that the flux density (0.602337) is positive when the arrows point towards the LHP, but those are the arrows of the Pólya vector plane $\overline{H}$. The arrows of vector field H would point towards the UHP.