

Needham, Chapter 11, Exercise 4(ii), p. 505.

G. Palmer, May 28, 2018, 11 pm PDT

4 Repeat parts (i) and (ii) of the previous question using $f(z) = z \csc^2 z$.

The previous question: Use a computer to draw the Pólya vector field of $f(z) = 1/[z \sin z]$ and thereby identify the locations and orders of the poles of $f(z)$. For each of the following choices of C , numerically estimate $\oint_C f(z) dz$ by making on-screen measurements of the vectors, then estimating the flux and circulation round C . In each case check your estimate by calculating the exact answer using residue theory.

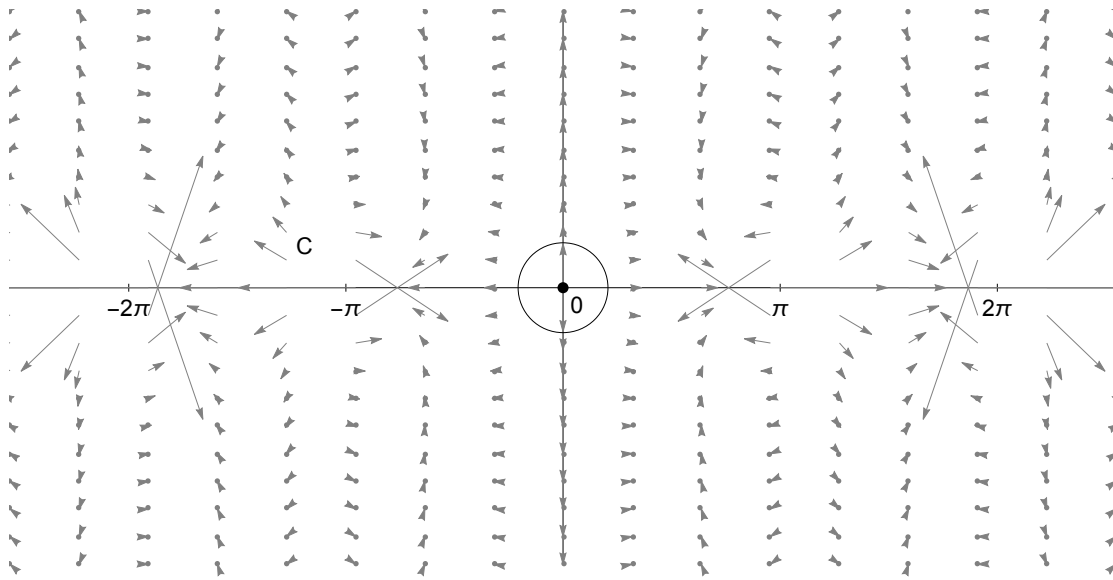
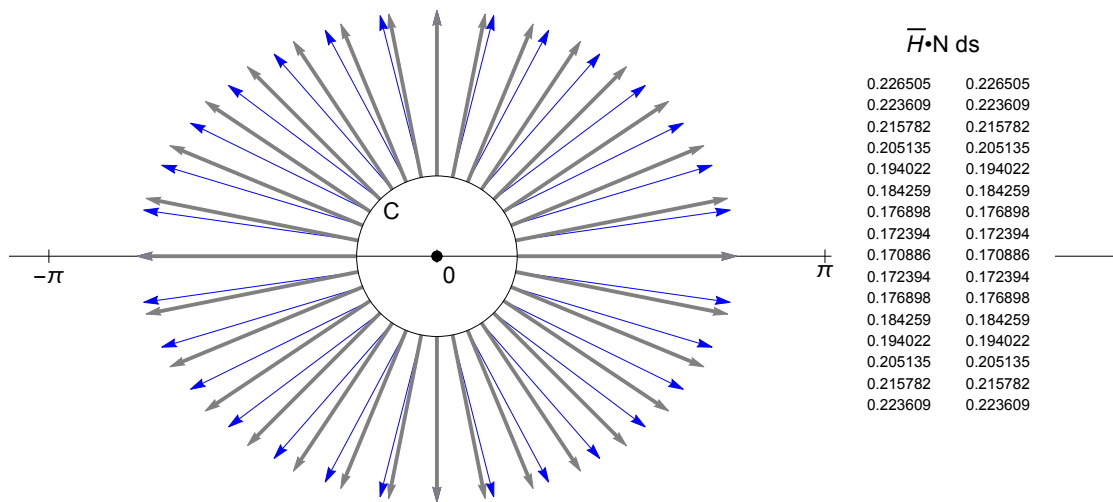
Figure 2a. Field vectors of $\text{Conj}(z \csc^2 z)$ 

Figure 4b. Pólya field (blue) of $H = z \csc^2(z)$ on C , $r < 1$
 projections onto normals $\bar{H} \cdot N$ (gray)
 $\sum (\bar{H} \cdot N) \, ds = 6.28319 \approx 2\pi$

(ii) Let C be a small circle centred at 0.

The Pólya vector field in Figure 2a appears to be symmetrical about both axes. The phase diagram in Figure 1 suggests poles at integer multiples of π , as they were for $1/[z \sin z]$, but here the $\sin(z)$ pole is replaced by the double pole $\sin^2(z)$, derived from $\csc^2(z)$. This double pole is multiplied by z ,

which has a root at zero. Then the quotient $z / \sin^2(z)$ must have the effect of a single pole such as $1/z$, which means that $\text{Res}[z \csc^2(z), 0] = \text{Res}[z / \sin^2(z), 0] = \text{Res}[1/z, 0] = 1$. This is supported by the residue calculations which follow discussion of the plots. The calculations of $(\Sigma H dz)$ from the field vectors on screen suggest the hypothesis that the residues are the same for $a = 0$ and $a = \pm n\pi$. That is, when we allow for the vanishing of the real number, corresponding to curl, we are left with $2\pi i \cdot \text{Res}[1/z, a] = 2\pi i$.

In Figure 2b , the blue arrows are Pólya field vectors on C. The values in the columns proceed anticlockwise starting from z at $(\overline{H} \cdot N) ds = 0.226505$. The plot suggests that curl vanishes on C because the field vector arrows are symmetrical on both the real line and the y axis.

Apply residue theory

$$z \csc^2 z = \frac{1}{z} + \frac{z}{3} + \frac{z^3}{15} \dots, \text{ so } \text{Res}[z \csc^2 z, 0] = 1$$

$$\oint_C f(z) dz = 2\pi i$$

$$W[\overline{f(z)}, C] = 0, \quad F[\overline{f(z)}, C] = 2\pi$$

This supports the calculations made from the plots.