

Needham, Chapter 11, Exercise 4(i), p. 505.

G. Palmer, June 28, 2018, 11:50 pm PDT

4 Repeat parts (i) and (ii) of the previous question using  $f(z) = z \csc^2 z$ .

**The previous question:** Use a computer to draw the Pólya vector field of  $f(z) = 1/[z \sin z]$  and thereby identify the locations and orders of the poles of  $f(z)$ . For each of the following choices of  $C$ , numerically estimate  $\oint_C f(z) dz$  by making on-screen measurements of the vectors, then estimating the flux and circulation round  $C$ . In each case check your estimate by calculating the exact answer using residue theory.

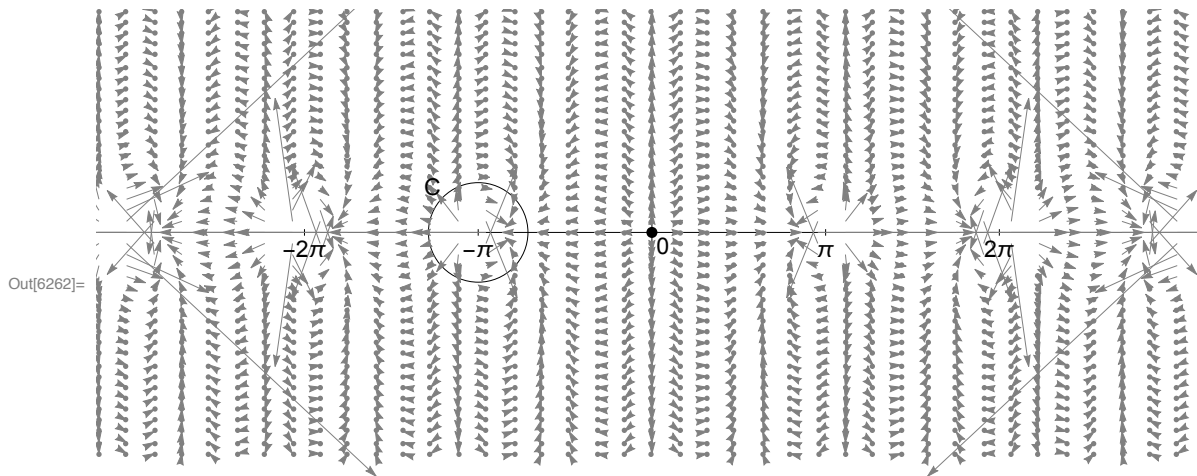
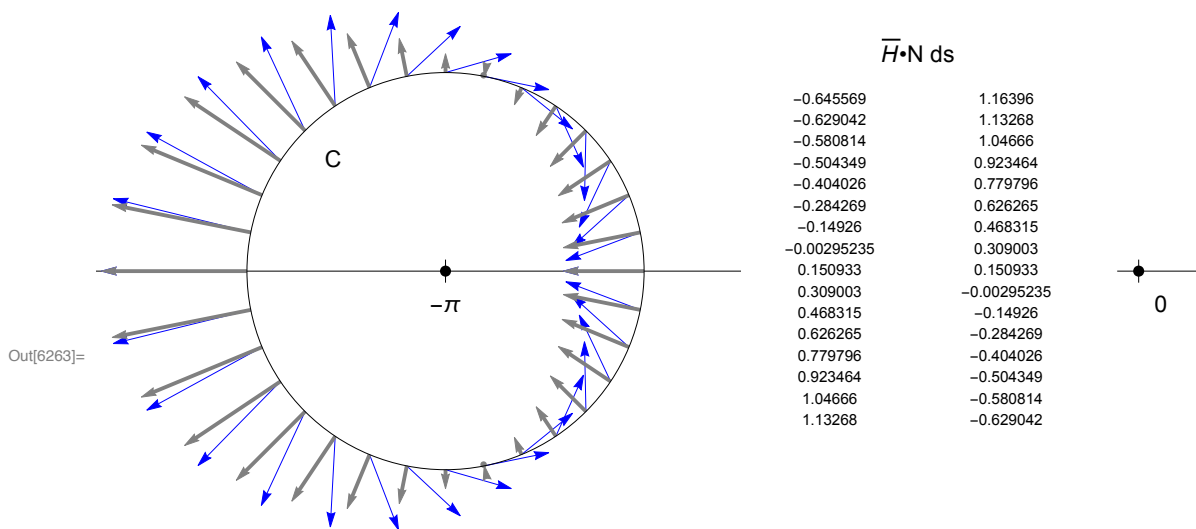
Figure 4a. Field vectors of  $\text{Conj}(z \csc^2(z))$ 

Figure 4b. Pólya field (blue) of  $H = z \csc^2(z)$  on  $C$ ,  $r < 1$   
 projections onto normals  $\overline{H} \cdot \mathbf{N}$  (gray)  
 $\sum (\overline{H} \cdot \mathbf{N}) \, ds = 6.28319 \approx 2\pi$

(i) Let  $C$  be a small circle centred at  $-\pi$ .

The vectors in plot 4a are drawn to 3/100 scale. The Pólya vector field in Figure 1a appears to be symmetrical about both axes. The vector arrows reveal the integer multiples of  $\pi$  to be the location of roots and poles. They are located at 0 and  $\pm n\pi$ , as they were for  $1/[z \sin z]$ , but here the  $\sin(z)$  pole is replaced by the double pole  $\sin^2(z)$ , derived from  $\csc^2(z)$ . This double pole is multiplied by  $z$ , which has a root at zero. Then the quotient  $z / \sin^2(z)$  must have the effect of a single pole such as

$1/z$ , which means that  $\text{Res}[z \csc^2(z), 0] = \text{Res}[z / \sin^2(z), 0] = \text{Res}[1/z, 0] = 1$ . This is supported by the residue calculations which follow discussion of the plots. The calculations of  $(\sum \overline{H} dz)$  from the field vectors on screen suggest the hypothesis that the residues are the same for  $a = 0$  and  $a = \pm n\pi$ . That is, when we allow for the vanishing of the real number, corresponding to curl, we are left with  $2\pi i \cdot \text{Res}[1/z, a] = 2\pi i$ .

In Figure 1b, the blue arrows are Pólya field vectors on  $C$ . The vectors are drawn to 1/10 scale. The values in the columns proceed anticlockwise starting from  $z$  at  $\overline{H} = 0.64557$ . The sum of the field vectors on  $C$  was taken using the usual five digits after the decimal point on all the data points. The plot suggests that curl vanishes on  $C$  because the field vector arrows are symmetrical on the real line.

**Apply residue theory** (See p. 435 (9) and the preceding equation and pp. 438-439, 4 Calculating Residues using Taylor Series.)

$$\oint_C f(z) dz = \oint_C z \csc^2 z dz$$

$$\csc z = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} \dots$$

$$\csc^2 z = \frac{1}{z^2} + \frac{1}{3} + \frac{z^2}{15} \dots$$

$$z \csc^2 z = \frac{1}{z} + \frac{z}{3} + \frac{z^3}{15} \dots, \text{ so } \text{Res}[z \csc^2 z, \pm n\pi \text{ and } 0] = 1$$

Then

$$\oint_C f(z) dz = 2\pi i \cdot 1 = 2\pi i$$

$$W[\overline{f(z)}, C] = 0, \quad F[\overline{f(z)}, C] = 2\pi$$

For (v) where  $C$  encloses  $\pi$  and  $2\pi$ ,

$$\text{Res}[z \csc^2 z, \pi] + \text{Res}[z \csc^2 z, 2\pi] = 1 + 1 = 2$$

$$\oint_C f(z) dz = 2\pi i \cdot 2 = 4\pi i$$

$$W[\overline{f(z)}, C] = 0, \quad F[\overline{f(z)}, C] = 4\pi$$