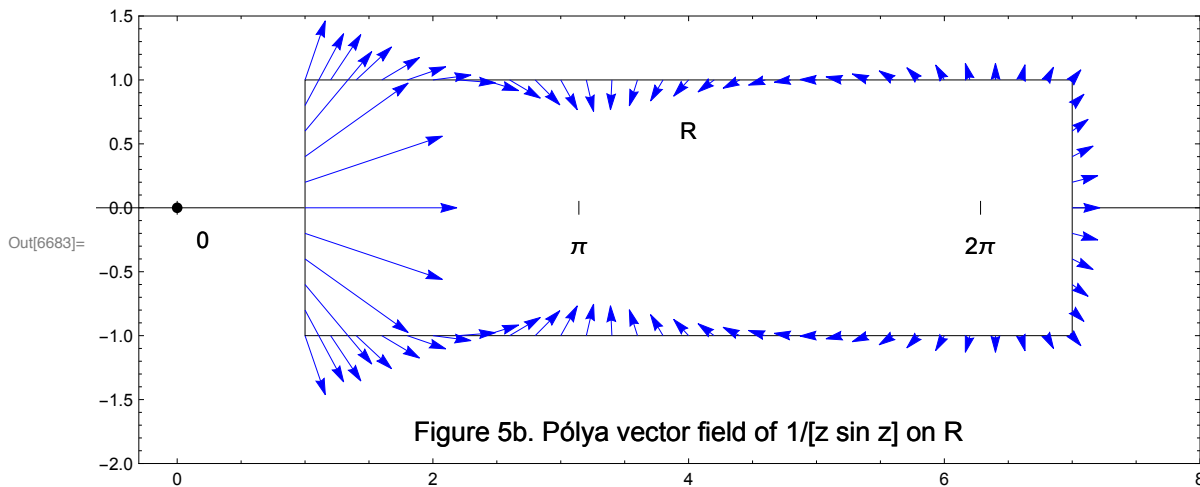
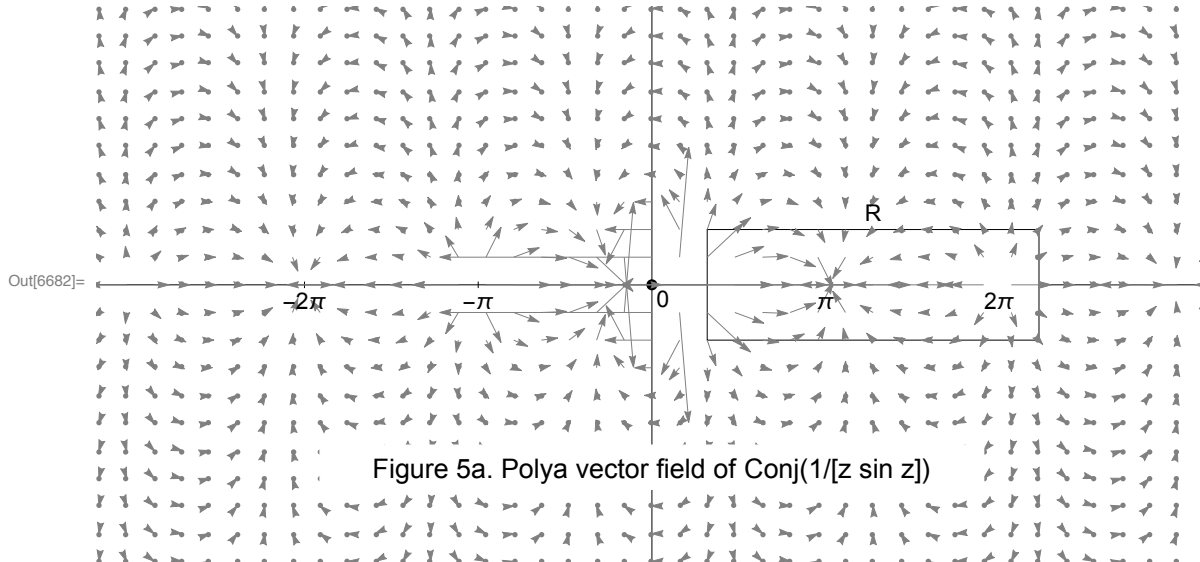


Needham, Chapter 11, Exercise 3(v), p. 505.

G. Palmer, June 15, 2018, 1:30 pm PDT

3 Use a computer to draw the Pólya vector field of $f(z) = 1/[z \sin z]$ and thereby identify the locations and orders of the poles of $f(z)$. For each of the following choices of C , numerically estimate $\oint_C f(z) dz$ by making on-screen measurements of the vectors, then estimating the flux and circulation round C . In each case check your estimate by calculating the exact answer using residue theory.



In[5501]:=

(i) Let C be the boundary of the rectangle $1 \leq x \leq 7, -1 \leq y \leq 1$.

Figure 5a illustrates the Pólya vector field and the location of R . It shows that the PVF encloses π and 2π and that the sum of the indices is 2, so there are two poles. In Figure 5b, one can see that $I[\overline{H}, \pi] + I[\overline{H}, 2\pi] = -1 - 1 = -2$, again showing two Pólya poles.

The symmetry of the vectors about the real line suggests that work vanishes on R . In finding flux density, the formula $\sum \mathbf{H} \cdot \mathbf{N} \, ds$ does not seem to be applicable to this problem, at least not as applied directly to R . On p. 485, subsection 4, we find that flux depends on winding number. With two poles, a single traversal of z on R would produce two winding numbers, and two integrals. Since the size and shape of a continuous contour enclosing a pole does not matter so long as the contour does not cross the pole, we need only consider number and location of the poles and we can use the results of (iii) and (iv). In this case, we can add the flux from π to the flux from 2π . The flux density from π is -2 , as calculated in (iii). The flux density from 2π is 1 , as calculated in (iv). So the sum of the flux densities on $R = -2 + 1 = -1$.

$\sum \mathbf{H} \, dz = f(z) \, dz$ on R gives $\oint_R f(z) \, dz = -i$, which supports the conclusion above and the results of applying residue theory. The calculation is not shown.

Apply residue theory

$$\oint_R f(z) \, dz = \oint_R \frac{1}{z \sin(z)} \, dz$$

R encloses two poles: π and 2π . Then by the General Residue Theorem (26), p. 419, and (9), p. 435, we have

$$\oint_L f(z) \, dz = 2\pi i \sum_n \nu[L, a_n] \operatorname{Res}[f, a_n]$$

Substitute $a_1 = \pi$ and $a_2 = 2\pi$. Then

$$\oint_C f(z) \, dz = 2\pi i (\nu[R, \pi] \operatorname{Res}[f(z), \pi] + \nu[R, 2\pi] \operatorname{Res}[f(z), 2\pi])$$

Applying the method of P/Q, let $P = \frac{1}{z}$ and $Q = \sin(z)$, so $Q' = \cos(z)$.

$$\operatorname{Residue}\left[\frac{1}{z \sin(z)}, \pi\right] = \frac{1}{\pi} \frac{1}{\cos(\pi)} = \frac{1}{\pi} \frac{1}{-1} = -\frac{1}{\pi}$$

$$\operatorname{Residue}\left[\frac{1}{z \sin(z)}, 2\pi\right] = \frac{1}{2\pi} \frac{1}{\cos(2\pi)} = \frac{1}{2\pi} \frac{1}{1} = \frac{1}{2\pi}$$

$$\oint_R f(z) \, dz = 2\pi i (\operatorname{Res}[f, \pi] + \operatorname{Res}[f, 2\pi])$$

$$= 2\pi i \left(-\frac{1}{\pi} + \frac{1}{2\pi}\right) = -i$$

Then by equating real and imaginary parts, $W[\overline{f(z)}, C] = 0$ and $F[\overline{f(z)}, C] = -1$.

Residue theory supports the answer obtained by plotting and improves its accuracy.

To provide a check on the answer from the plots and the residue theory, we integrate on the four sides of R using the Fundamental Theorem of Calculus.

$$\int_R f(z) dz = \int_N f(z) dz + \int_W f(z) dz + \int_S f(z) dz + \int_E f(z) dz$$

$$\int_N \frac{1}{z \sin(z)} dz$$

$$z(t) = (7+i)-7t, \text{ where } 0 \leq t \leq 1$$

$$v(t) = -7$$

We use numerical integration in *Mathematica*.

$$\int_N \frac{1}{z \sin(z)} dz = -7 \int_0^1 \frac{1}{((7+i)-7t) \sin((7+i)-7t)} dt = -0.259885+0.0322067 i$$

$$\int_W \frac{1}{z \sin(z)} dz$$

$$z(t) = (1+i)-2it, \text{ where } 0 \leq t \leq 1$$

$$v(t) = -2i$$

$$\int_W \frac{1}{z \sin(z)} dz = -2i \int_0^1 \frac{1}{((1+i)-2it) \sin((1+i)-2it)} dt = 0.-1.35558 i$$

$$\int_S \frac{1}{z \sin(z)} dz$$

$$z(t) = (1-i)+6t, \text{ where } 0 \leq t \leq 1$$

$$v(t) = 6$$

$$\int_S \frac{1}{z \sin(z)} dz = 6 \int_0^1 \frac{1}{((1-i)+6t) \sin((1-i)+6t)} dt = 0.259885+0.0322067 i$$

$$\int_E \frac{1}{z \sin(z)} dz$$

$$z(t) = (7-i) + 2it, \text{ where } 0 \leq t \leq 1$$

$$v(t) = 2i$$

$$\int_E \frac{1}{z \sin(z)} dz = 2i \int_0^1 \frac{1}{((7-i)+2it) \sin((7-i)+2it)} dt = -1.38778 \times 10^{-17} + 0.291171 i$$

Then

$$\int_R f(z) dz = -i$$

Integration using the FTC on the sides of R supports the answers from plotting and residue theory. Residue theory provides the easiest path to the most accurate answer.

We now have several ways to calculate $\int_R f(z) dz$ on these poles.

$\Sigma H dz$

$\Sigma (\overline{H} \cdot T + \overline{H} \cdot N) ds$

The Fundamental Theorem of Calculus

Residue Theory