

Needham, Chapter 11, Exercise 3(iii), p. 505.

G. Palmer, June 15, 2018, 9:20 am PDT

3 Use a computer to draw the Pólya vector field of $f(z) = 1/[z \sin z]$ and thereby identify the locations and orders of the poles of $f(z)$. For each of the following choices of C , numerically estimate $\oint_C f(z) dz$ by making on-screen measurements of the vectors, then estimating the flux and circulation round C . In each case check your estimate by calculating the exact answer using residue theory.

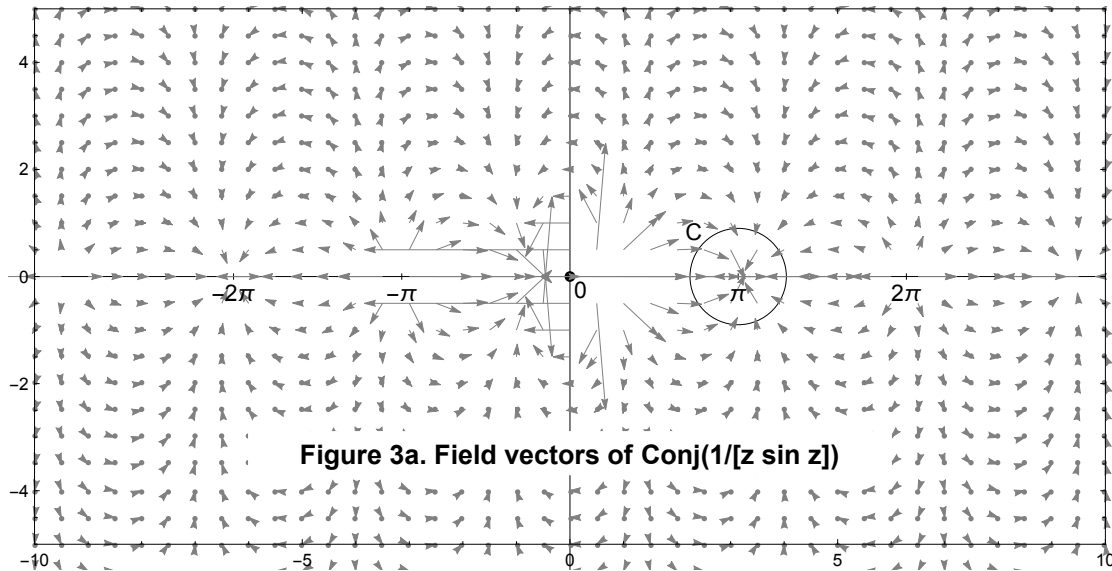


Figure 3a. Field vectors of $\text{Conj}(1/[z \sin z])$

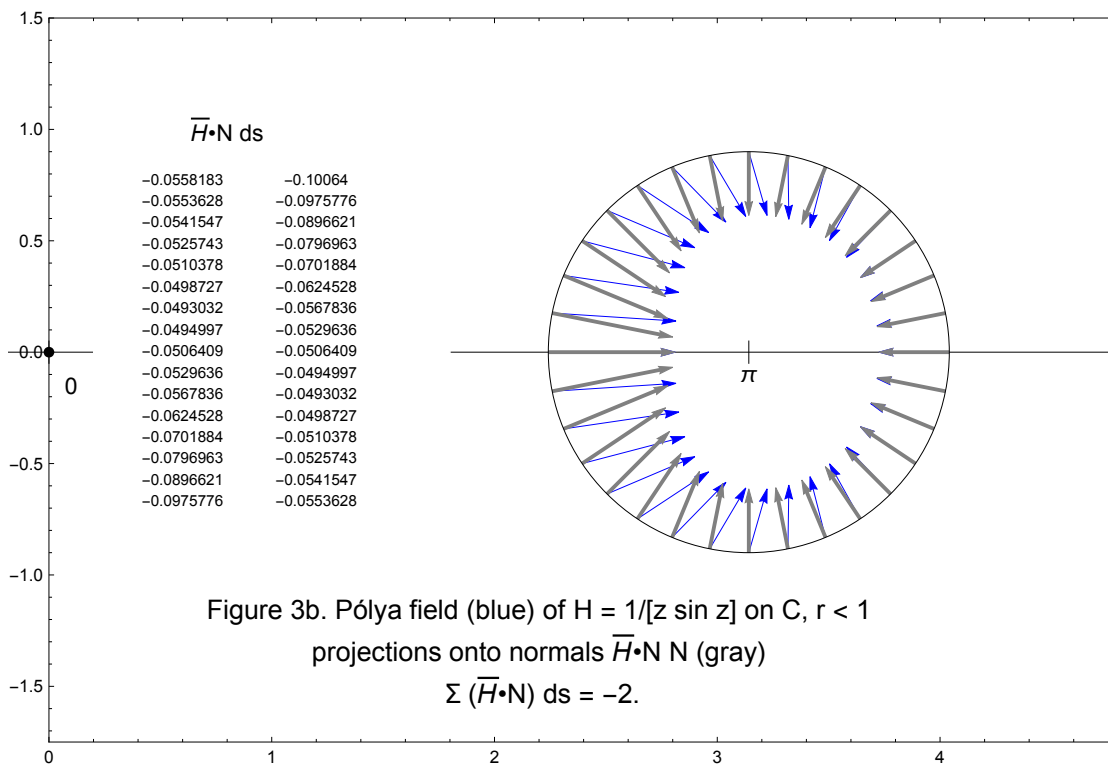
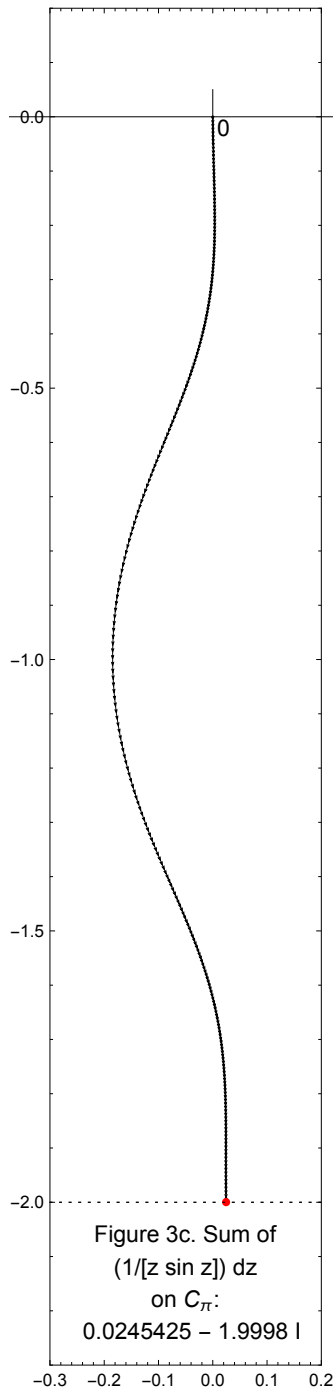


Figure 3b. Pólya field (blue) of $H = 1/[z \sin z]$ on C , $r < 1$
projections onto normals $\bar{H} \cdot N \, ds$ (gray)

$$\Sigma (\bar{H} \cdot N) \, ds = -2.$$



(i) Let C be a small circle centred at π .

Figure 3a is like 1a, except the centre of C has moved to π . The arrows in the neighborhood of π suggest that π is a Pólya sink.

Figure 3b shows that the Pólya vectors (blue) have positive rotation, indicating that π is a pole of

order 1. An estimation of $F[\overline{H}, C]$ using the Pólya vectors gives $\Sigma \overline{H} \cdot \mathbf{N} \, ds = -2$. Inspection of the vectors suggests that curl vanishes given enough data points because of the symmetry about the real line.

Figure 3c provides a visual illustration that $\Sigma H \, dz$ on 256 points of C . The result approximates $\oint_C f(z) \, dz = -2i$. This is another indication that curl vanishes, and the result agrees with the result from on-screen measurement and residue theory.

Apply *residue theory*

$$\oint_C f(z) \, dz = \oint_C \frac{1}{z \sin(z)} \, dz$$

Applying the method of P/Q, let $P = \frac{1}{z}$ and $Q = \sin(z)$, so $Q' = \cos(z)$.

$$\text{Residue}\left[\frac{1}{z \sin(z)}, \pi\right] = \frac{1}{\pi} \frac{1}{\cos(\pi)} = \frac{1}{\pi} \frac{1}{-1} = -\frac{1}{\pi}$$

$$\oint_C f(z) \, dz = 2\pi i \, \text{Res}[f, \pi] = 2\pi i \left(-\frac{1}{\pi}\right) = -2i$$

Then $W[\overline{f}(z), C] = 0$ and $F[\overline{f}(z), C] = -2$.