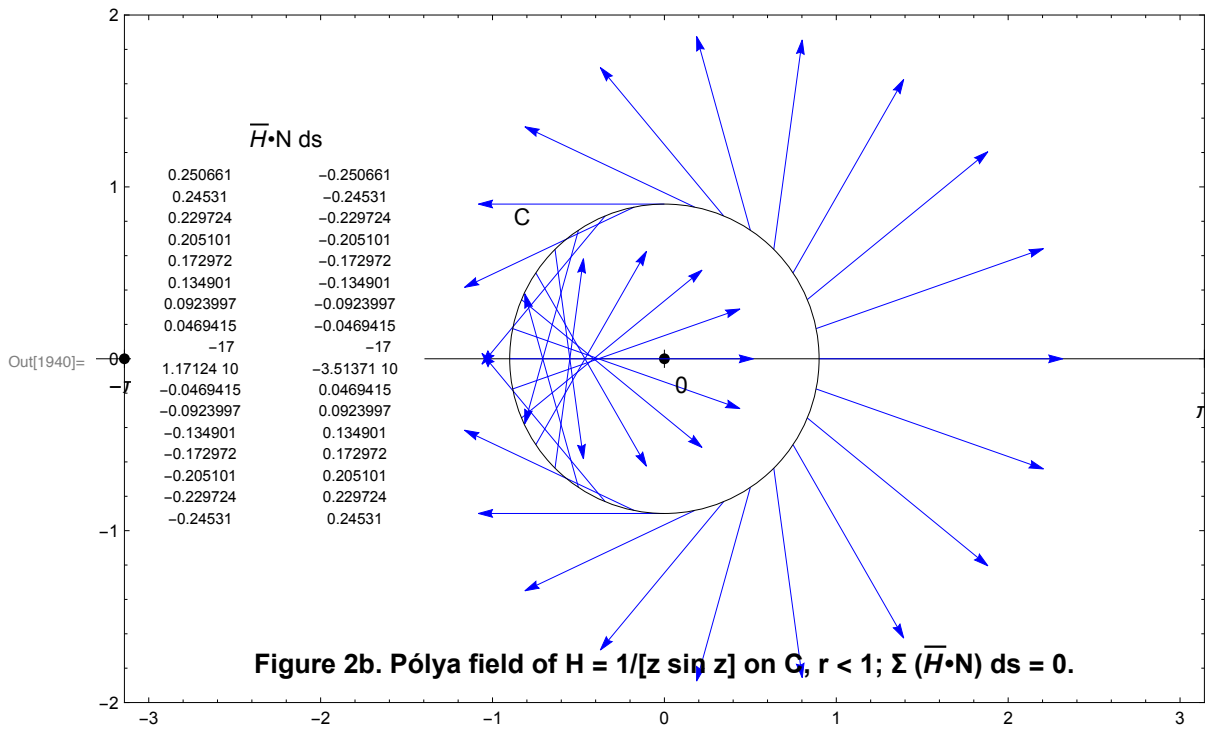
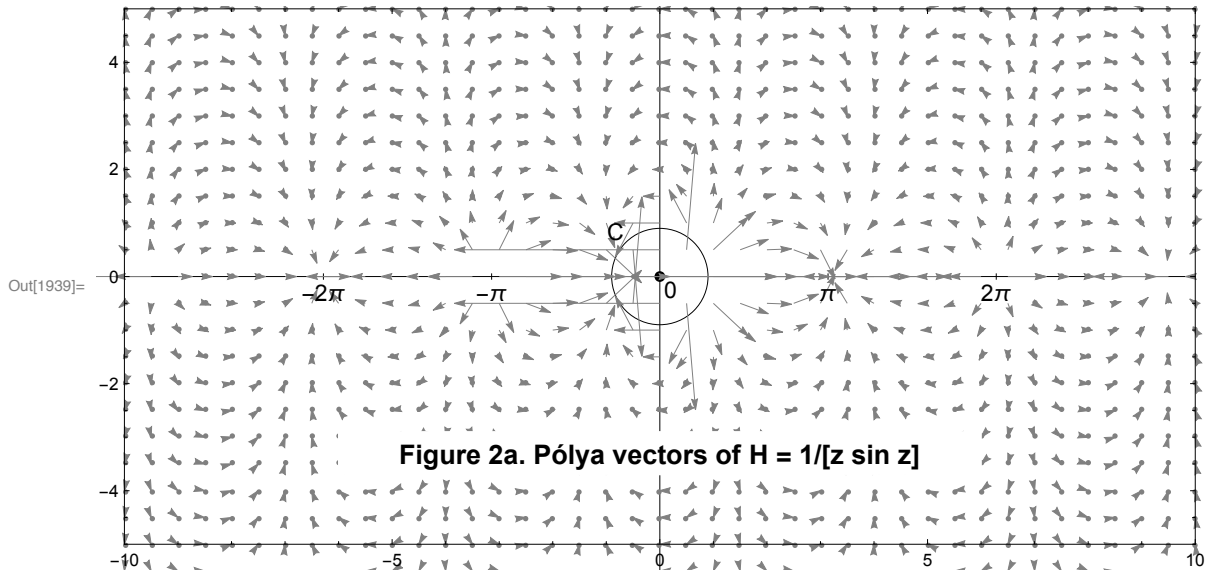
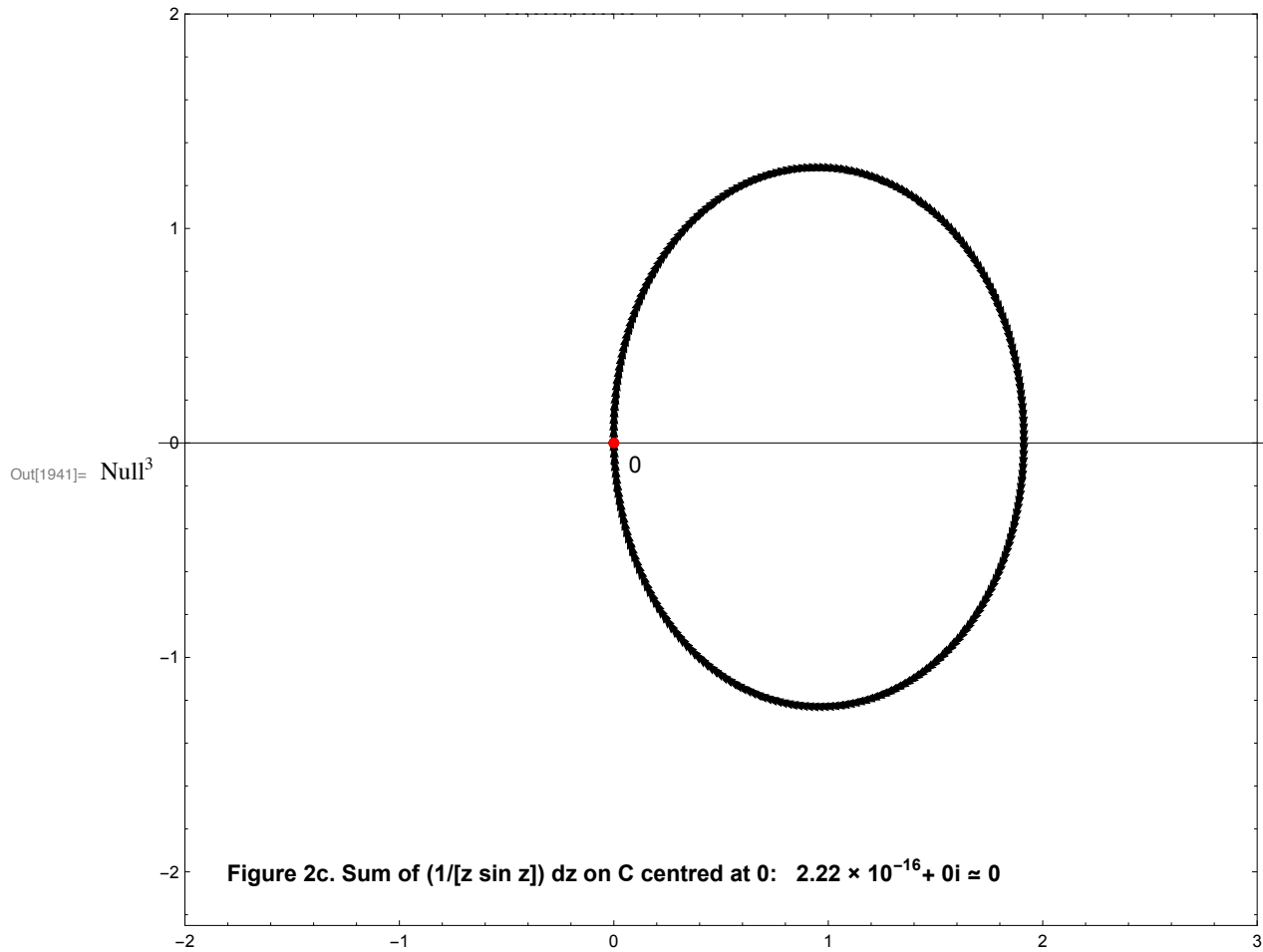


Needham, Chapter 11, Exercise 3 (ii), p. 505.

G. Palmer, June 14, 2018, 10 pm PDT

3 Use a computer to draw the Pólya vector field of $f(z) = 1/[z \sin z]$ and thereby identify the locations and orders of the poles of $f(z)$. For each of the following choices of C , numerically estimate $\oint_C f(z) dz$ by making on-screen measurements of the vectors, then estimating the flux and circulation round C . In each case check your estimate by calculating the exact answer using residue theory.





In[802]:=

(ii) Let C be a small circle centred at 0.

Figure 2a, is the same as 1a, except the circle C has been moved to the turbulent region near 0.

The rotation of $f(z)$ on C is most easily seen in Figure 2b, which shows on close examination that $\mathcal{I}[\overline{f(z)}, 0] = 2$. Hence, zero is a pole of order 2.

Figure 2b shows that both flux density and curl vanish on C . One would guess from the symmetry on the real line that curl would vanish. Here, flux density also vanishes. The column header does not reproduce the bar over the H , but it should be there.

Figure 2c provides a visual illustration that summing 256 instances of $H dz = f(z) dz$ on C gives $\oint_C f(z) dz = 0$, which confirms the result of the on-screen measurements and the results of residue theory.

Apply residue theory

$$\oint_C f(z) dz = \oint_C \frac{1}{z \sin(z)} dz$$

The method of P/Q' can not be applied here to find the residue, as the result would be $\infty/0$. Instead, we note that $\frac{1}{z \sin(z)}$ can be rewritten $\csc(z)/z$, and we find the Laurent series:

$$f(z) = \frac{1}{z} \left(\frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \dots \right) = \left(\frac{1}{z^2} + \frac{1}{6} + \frac{7z^2}{360} + \dots \right)$$

Since there is no $\frac{1}{z}$ term, $\text{Res}[f, 0] = 0$.

$$\oint_C f(z) dz = 2\pi i \text{Res}[f, 0] = 0$$

Then $W[\overline{f(z)}, C] = 0$ and $F[\overline{f(z)}, C] = 0$.