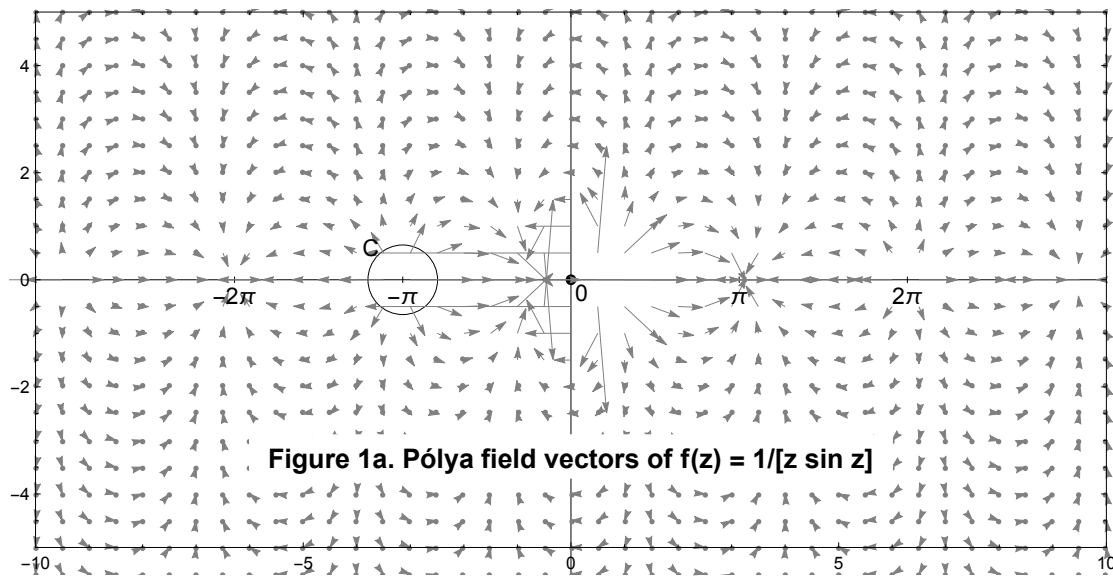


Needham, Chapter 11, Exercise 3(i), p. 505.

G. Palmer, June 19, 2018, 9 am PDT

3 Use a computer to draw the Pólya vector field of $f(z) = 1/[z \sin z]$ and thereby identify the locations and orders of the poles of $f(z)$. For each of the following choices of C , numerically estimate $\oint_C f(z) dz$ by making on-screen measurements of the vectors, then estimating the flux and circulation round C . In each case check your estimate by calculating the exact answer using residue theory.



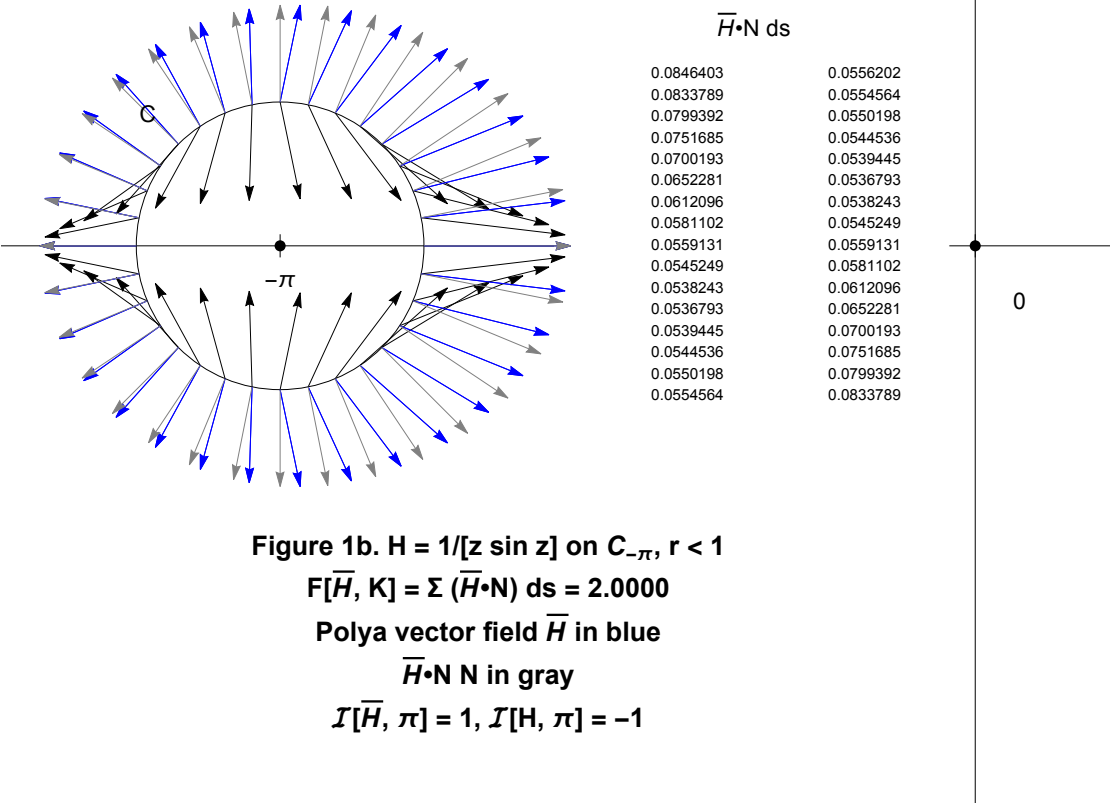
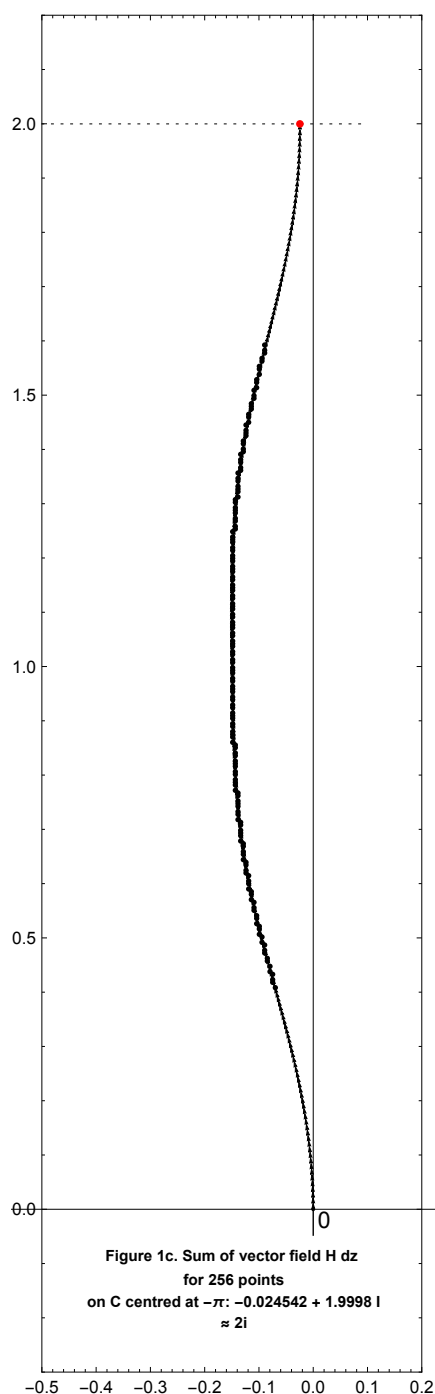


Figure 1b. $H = 1/[z \sin z]$ on $C_{-\pi}$, $r < 1$
 $F[\bar{H}, K] = \Sigma (\bar{H} \cdot N) ds = 2.0000$
Polya vector field $\bar{H}$ in blue
 $\bar{H} \cdot N$ in gray
 $\mathcal{I}[\bar{H}, \pi] = 1$, $\mathcal{I}[H, \pi] = -1$



(i) Let C be a small circle centred at $-\pi$.

On screen analysis and measurements

In Figure 1a, the Polya vector arrows point to poles located at multiples of π , as one would expect from the formula for $H = f(z)$. One can see that the field vectors increase in length as z approaches

0, suggesting that 0 is also a pole. It also appears that if $n > 0$ is odd, then $n\pi$ is a sink, and if $n > 0$ is even, then $n\pi$ is a source. This assignment is reversed for $n < 0$. Zero is a pole, but it appears to be neither source nor sink. By observing the rotation of the field vectors on the circle (Figure 1b), we conclude that the index $\mathcal{I}[f(z), -\pi] = -1$, showing that $-\pi$ is a pole of order 1. The index $\mathcal{I}[\overline{f}(z), -\pi] = 1$ (blue arrows), showing the opposite rotation of the conjugate. We conclude that $-\pi$ is a pole of order 1 if the index of the PVF is 1.

In Figure 1b, the black arrows are field vectors on C . The blue arrows are the Pólya field vectors. The gray arrows are the projections $(\overline{H} \cdot N) N$. The values in the columns for $(\overline{H} \cdot N) ds$ proceed anticlockwise starting from z on the real line where $\overline{H} \cdot N ds = 0.085$. We want to apply (11): $\int_K H(z) dz = W[\overline{H}, K] + i F[\overline{H}, K]$.

The plot suggests that curl vanishes on C because the field vector arrows (and the Pólya vector arrows) are symmetrical about the real line. Then we know that $W[\overline{H}, K] = 0$, so we need only calculate the flux density $F[\overline{H}, K]$. From p. 473, we find $F[X, K] = \int_K (X \cdot N) ds$. Then

$$F[\overline{H}, K] = \int_K (\overline{H} \cdot N) ds$$

$$\int_K H(z) dz = i \int_K (\overline{H} \cdot N) ds \approx i \Sigma(\overline{H} \cdot N) ds$$

Figure 2 shows that this sum for 32 points $\Sigma(\overline{H} \cdot N) ds = 2.0000$, which seems remarkable accuracy for so few sample points. This is the exact number obtained with residue theory presented below. Figure 1c was not required for the problem, but it shows that summing 256 instances of $H dz$ equals $-0.0245425 + 1.9998i$, which is close to the estimate of $\oint_C f(z) dz = 2i$ from residue theory and confirms the on screen measurements on the Pólya vector field. The Pólya field resulted in better accuracy than using $H dz$, which is not surprising, since dz was calculated as a chord to approximate ds , so the value would be slightly low.

Apply residue theory

$$\oint_C f(z) dz = \oint_C \frac{1}{z \sin(z)} dz$$

Using the guidance on pp. 434-436, let $P(z) = 1/z$ and $Q(z) = \sin(z)$. Then $Q' = \cos(z)$.

$$\begin{aligned} \text{Res}[f, -\pi] &= P/Q' = \frac{1}{z} \frac{1}{\cos z} = \frac{1}{-\pi} \frac{1}{\cos(-\pi)} = \frac{1}{-\pi} \frac{1}{-1} = 1/\pi \\ \oint_C f(z) dz &= 2\pi i \text{Res}[f, -\pi] = 2\pi i / \pi = 2i \end{aligned}$$

Since $\oint_C f(z) dz = W[\overline{f}(z), C] + i F[\overline{f}(z), C]$ from (11) p. 483, we can set real and imaginary parts equal. Then $W[\overline{f}(z), C] = 0$ and $F[\overline{f}(z), C] = 2$.